UNIVERSITY OF SOUTH CAROLINA

CSCE 590 INTRODUCTION TO IMAGE PROCESSING

Skeleton

Frequency domain

Correlation

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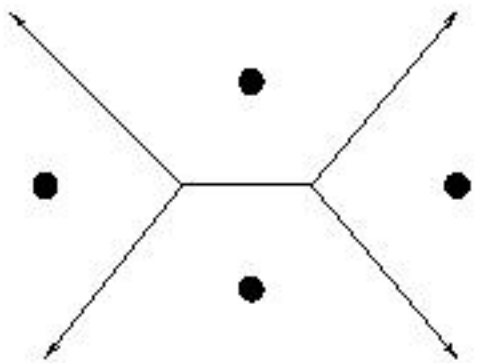
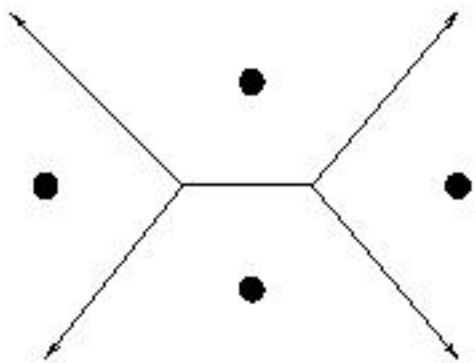
Voronoi diagrams



These line segments make up the **Voronoi diagram** for the four points shown here.

Solves the “Post Office Problem”

Voronoi diagrams



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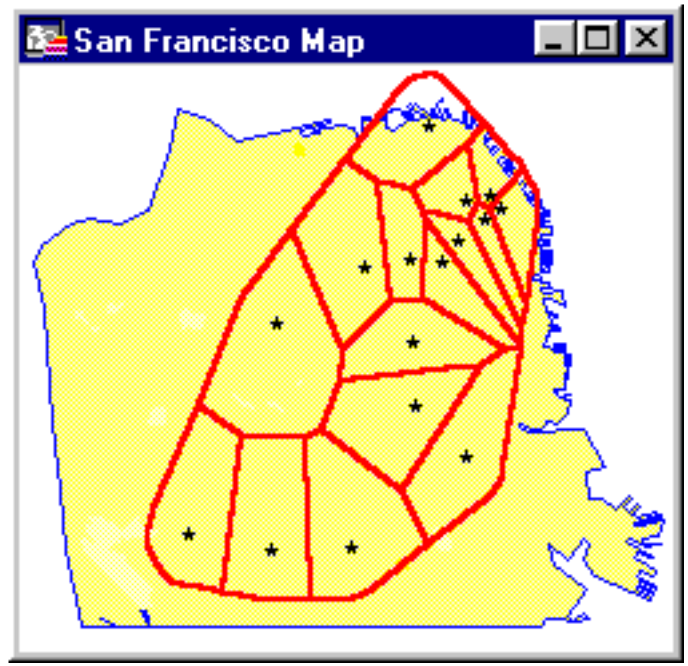
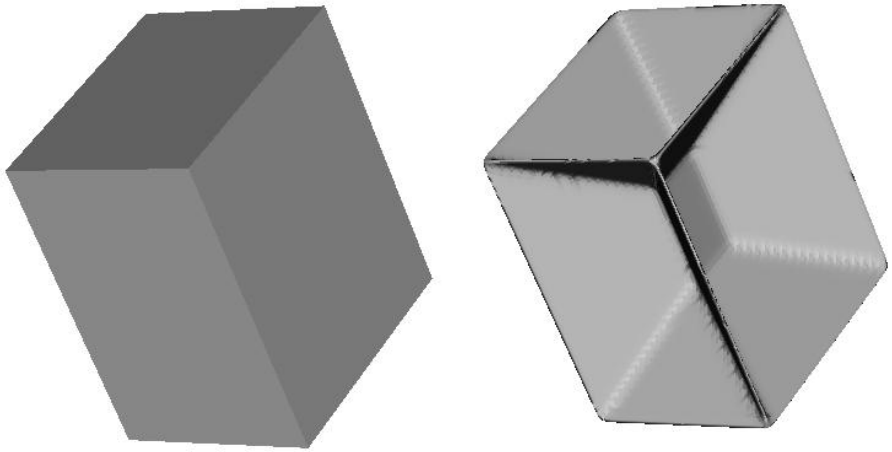
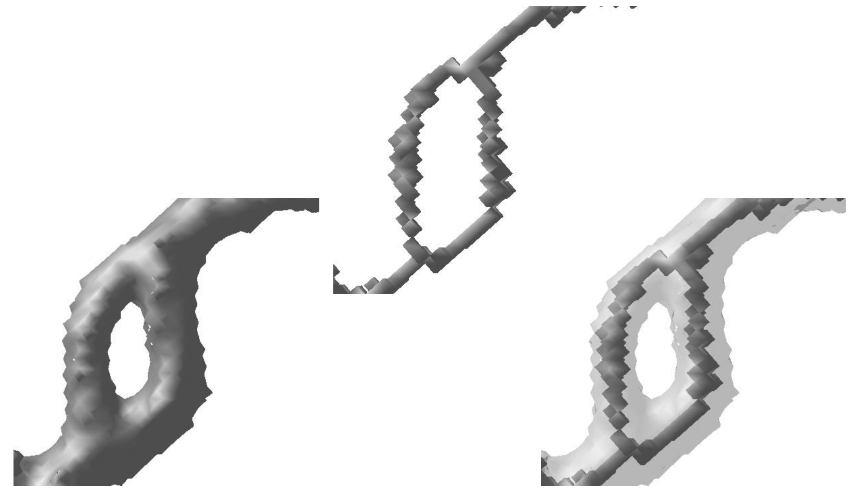


or, perhaps, more important problems…

Voronoi applications

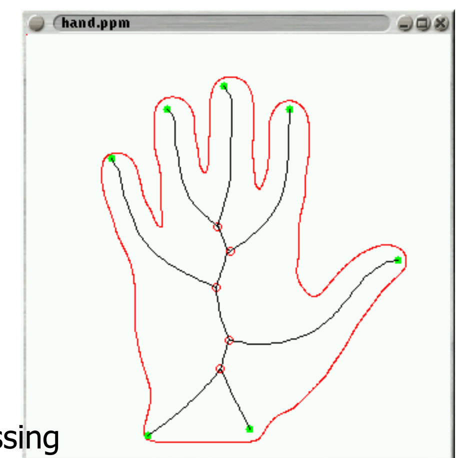




A retraction of a 3d object
== "*medial surface*"

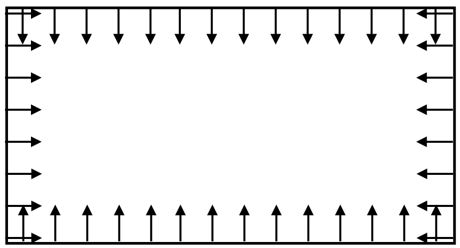
what?

Skeletonizations resulting from constant-speed curve evolution

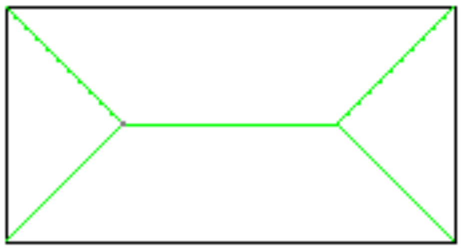


in 2d, it's called a *medial axis*

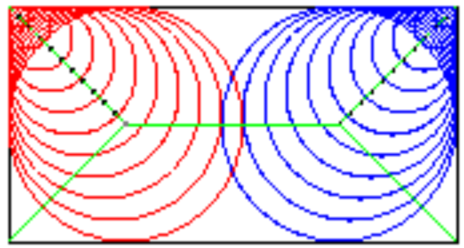
skeleton ⟷ shape



curve evolution



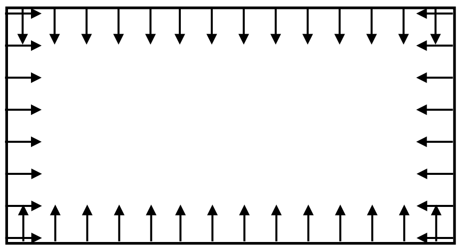
where wavefronts collide



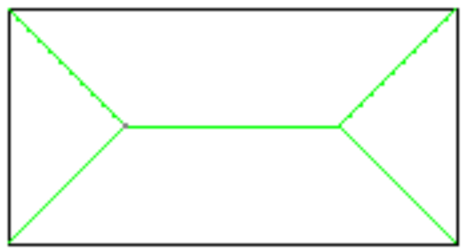
centers of maximal disks

again reduces a 2d (or higher) problem to a question about graphs...

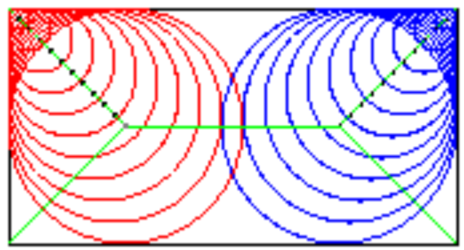
skeleton ⟷ shape



curve evolution

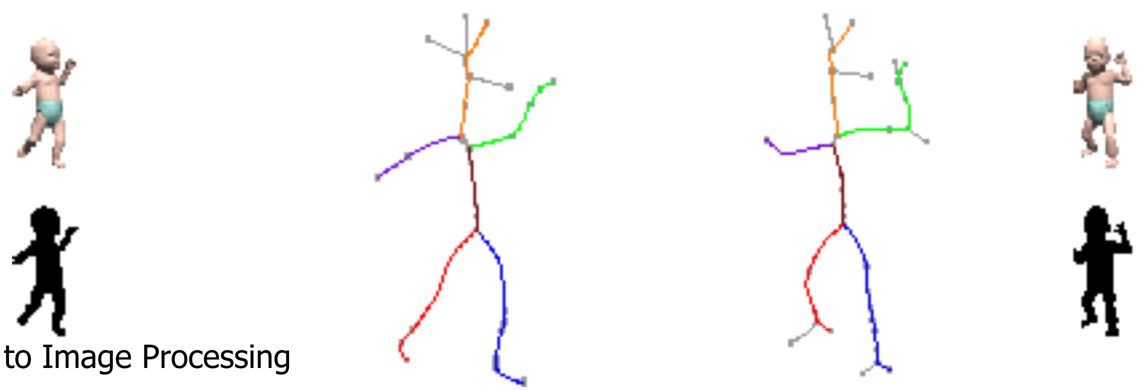


where wavefronts collide



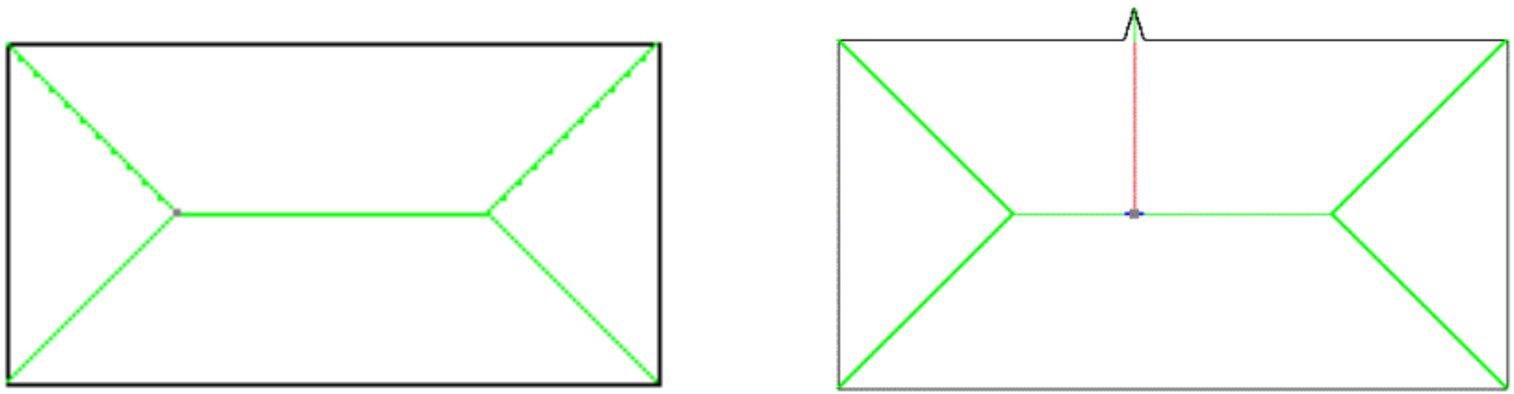
centers of maximal disks

again reduces a 2d (or higher) problem to a question about graphs...



graph matching

Problems



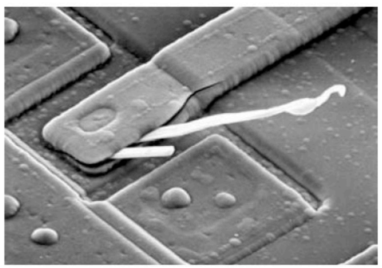
The skeleton is sensitive to small changes in the object's boundary.

- graph isomorphism (and lots of other graph questions) : NP-complete

1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	1
1	2	3	3	3	3	3	3	2	1
1	2	3	4	4	4	4	3	2	1
1	2	3	4	4	4	4	3	2	1
1	2	3	3	3	3	3	3	2	1
1	2	2	2	2	2	2	2	2	1
1	1	1	1	1	1	1	1	1	1

Why We Need Fourier Transform

- Filtering in frequency domain



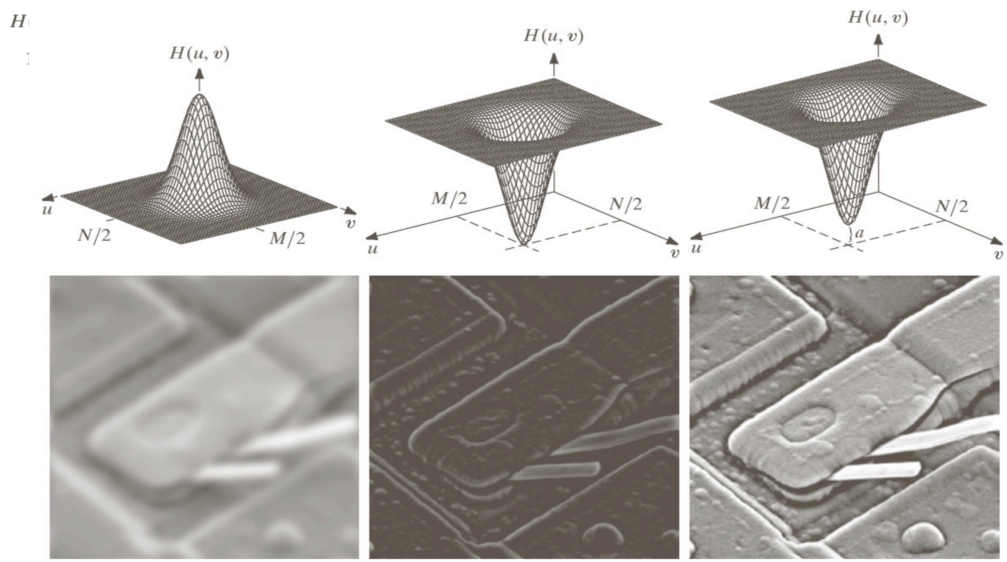
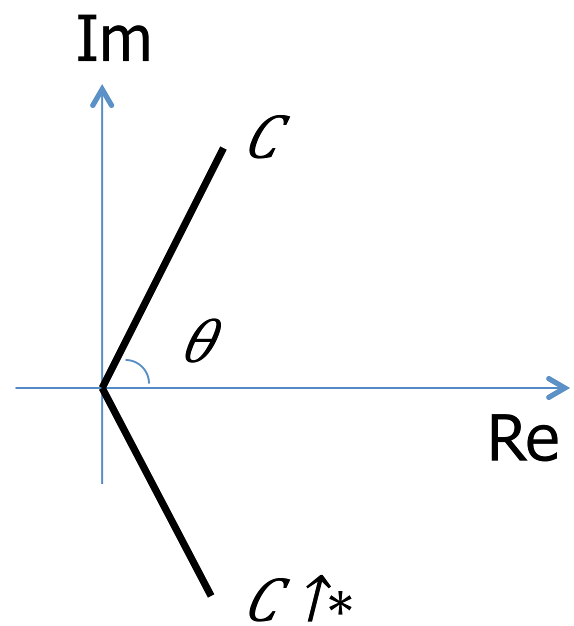


Image smoothing Edge Image sharpening

- Efficient computation for convolution

Preliminary Concepts



- Complex number $C = R + jI \quad j = \sqrt{-1}$

- Conjugate $C^* = R - jI$

- Polar coordinate representation

$$C = |C|(\cos\theta + j\sin\theta)$$

- Euler's formula $|C| = \sqrt{R^2 + I^2}, \qquad \theta = \arctan(I/R)$

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad C = |C|e^{j\theta}$$

Concept of Fourier Series And Transforms

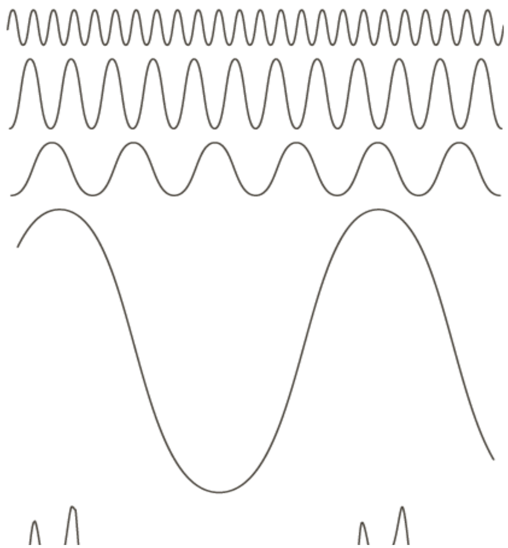


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier series: any periodic function can be represented by a discrete weighted sum of sines and cosines

Fourier transform: an arbitrary function with finite duration (non-periodic function) can be expressed by a weighted integrals of sines and cosines

Fourier transform is more general!

Fourier Series

- $f(t)$ is a continuous function with period T, we have

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{j2\pi nt}{T}}$$

Coefficient — Discrete frequency

- where $c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-\frac{j2\pi nt}{T}} dt, \quad n = 0, \pm 1, \pm 2, \ldots$

https://en.wikipedia.org/wiki/Fourier_transform#/media/File:Fourier_transform_time_and_frequency_domains_(small).gif

Fourier Transform in 1D

- $\boldsymbol{f(t)}$ is an arbitrary non-periodic function and can be represented by

$$f(t)=\int_{-\infty}^{\infty}F(\mu)e^{j2\pi\mu t}d\mu$$

Coefficient

Continuous frequency

where

$$F(\mu)=\int_{-\infty}^{\infty}f(t)e^{-j2\pi\mu t}dt$$

Fourier series

Discrete frequency

$$f(t)=\sum_{n=-\infty}^{+\infty}c_n e^{\frac{j2\pi nt}{T}}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-\frac{j2\pi nt}{T}}dt$$

Fourier Transform in 1D

- Spatial domain → Frequency domain

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$ Forward transform

- Frequency domain → Spatial domain

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$ Inverse transform

Fourier transform pair

Basic Properties of FT

- Linearity $h(t) = af(t) + bg(t) \leftrightarrow H(\mu) = aF(\mu) + bG(\mu)$
- Translation $h(t) = f(t - t_0) \leftrightarrow H(\mu) = e^{-j2\pi t_0 \mu} F(\mu)$
- Modulation $h(t) = e^{j2\pi\mu_0 t} f(t) \leftrightarrow H(\mu) = F(\mu - \mu_0)$
- Scaling $h(t) = f(at) \leftrightarrow H(\mu) = \frac{1}{|a|} F(\frac{\mu}{a})$
- Conjugation $h(t) = f^*(t) \leftrightarrow H(\mu) = F^*(-\mu)$
- Symmetry

$$f(t) \leftrightarrow F(\mu) \Rightarrow F(t) \leftrightarrow f(-\mu)$$

FT of Simple Functions

$$f(t)=\begin{cases} A & -w/2 \le t \le w/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mu)=\frac{A}{\pi\mu}\sin\pi w\mu = Aw\frac{\sin\pi w\mu}{\pi w\mu} = Aw\, sinc(\pi w\mu)$$

FT of a Rectangle Function

Rectangle function → Sinc function

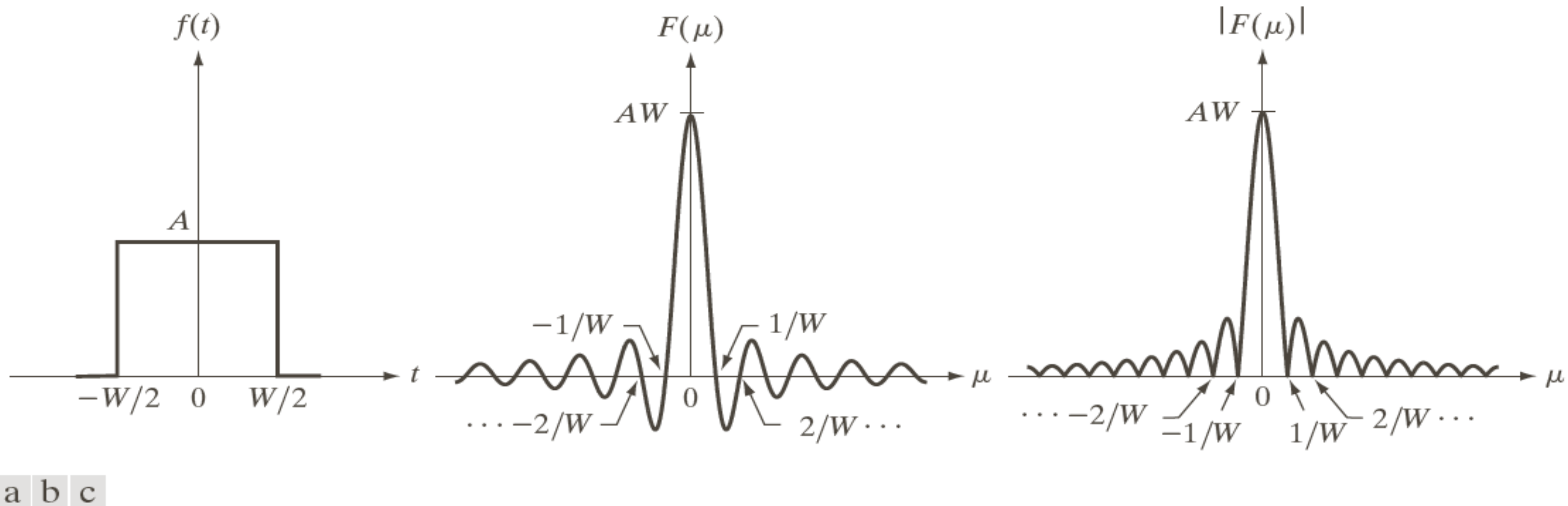


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Slides courtesy of Prof. Yan Tong

Continuous Impulses and Sifting Property

Unit impulse

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Sifting property

$$\int_{-\infty}^{\infty} \delta(t)g(t)dt = g(0)$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)g(t)dt = g(t_0)$$

The value of function at the impulse location

Discrete Impulses and Sifting Property

Unit impulse

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \sum_{x=-\infty}^{+\infty} \delta(x) = 1$$

Sifting property

$$\sum_{x=-\infty}^{\infty} \delta(x) g(x) = g(0)$$

$$\sum_{x=-\infty}^{\infty} \delta(x - x_0) g(x) = g(x_0)$$

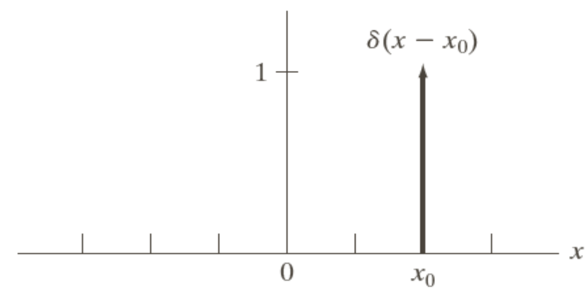


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

FT of an Impulse

$$\delta(t) \Leftrightarrow ?$$

$$\delta(t - t_0) \Leftrightarrow ?$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Proof with

- sifting property

$$\int_{-\infty}^{\infty} \delta(t - t_0) g(t) dt = g(t_0)$$

- translation property

$$h(t) = f(t - t_0) \Leftrightarrow H(\mu) = e^{-j2\pi t_0 \mu} F(\mu)$$

FT of an Impulse

$$\delta(t) \Leftrightarrow F(\mu) = 1$$

$$\delta(t - t_0) \Leftrightarrow F(\mu) = e^{-j2\pi\mu t_0}$$

FT of an Impulse

$$e^{j2\pi t_0 t} \leftrightarrow ?$$

$$F(e^{j2\pi t_0 t}) = \delta(\mu - t_0)$$

Symmetry property

$$f(t) \leftrightarrow F(\mu) \Rightarrow F(t) \leftrightarrow f(-\mu)$$

$$\delta(t - t_0) \leftrightarrow F(\mu) = e^{-j2\pi u t_0}$$

$$F(e^{-j2\pi t_0 t}) = \delta(-\mu - t_0)$$

$$F(e^{j2\pi t_0 t}) = \delta(\mu - t_0)$$

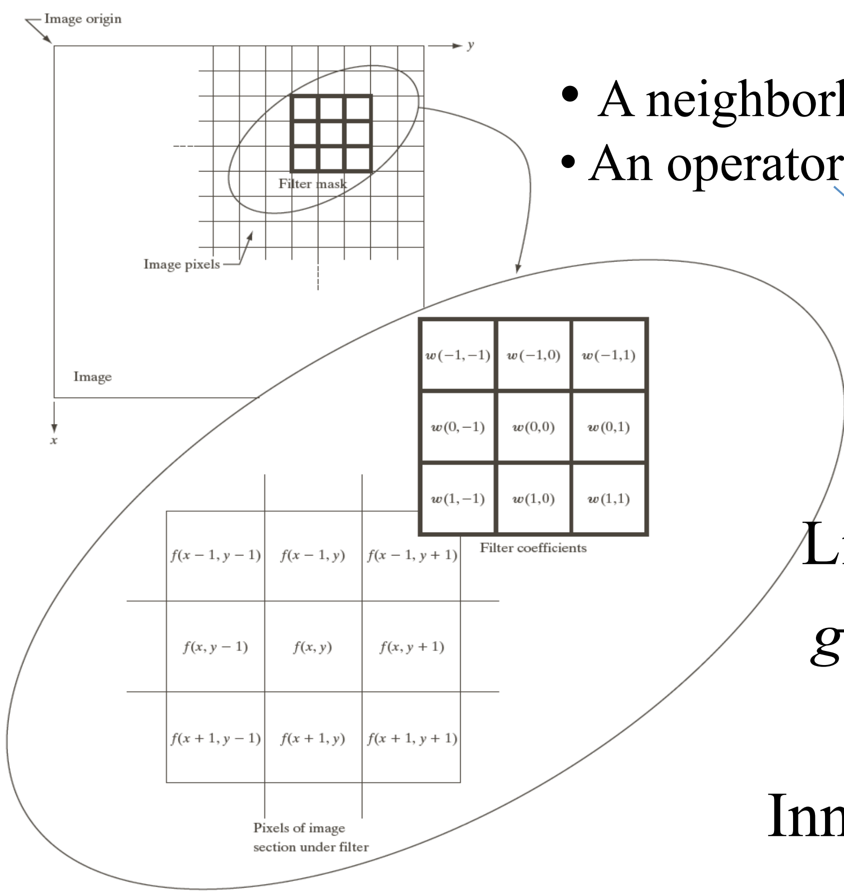
Scaling property

$$h(t) = f(at) \leftrightarrow H(\mu) = \frac{1}{|a|} F(\frac{\mu}{a})$$

MATLAB

```
➢ B1=zeros(15,15);
➢ B1(8,1:15)=1/15;
➢ Gb1=conv2(G,B1);
➢ F1=fft2(Gb1);
➢ Figure(1);
➢ imshow(log(fftshift(abs(F1)))/(max(max(log(abs(F1))))));
➢ B2=eye(15,15)/15;
➢ Gb2=conv2(G,B1);
➢ F2=fft2(Gb2);
➢ imshow(log(fftshift(abs(F2)))/(max(max(log(abs(F2))))));
➢ B1l=zeros(400,400);
➢ B1l(1:15,1:15)=B1;
➢ Fb1=fft2(B1l);
➢ figure(6); imshow((fftshift(abs(Fb1)))/(max(max((abs(Fb1))))));
➢ B2l=zeros(400,400);
➢ B2l(1:15,1:15)=B2;
➢ Fb2=fft2(B2l);
➢ figure(6); imshow((fftshift(abs(Fb2)))/(max(max((abs(Fb2))))));
```

Fundamentals of Spatial Filtering



- A neighborhood
- An operator with the same size: linear/nonlinear

Note: Each element in w will visit every pixel in the image just once.

Linear spatial filtering:

$$g(x,y)=\sum_{s=-a}^{a}\sum_{t=-b}^{b}w(s,t)f(x+s,y+t)$$

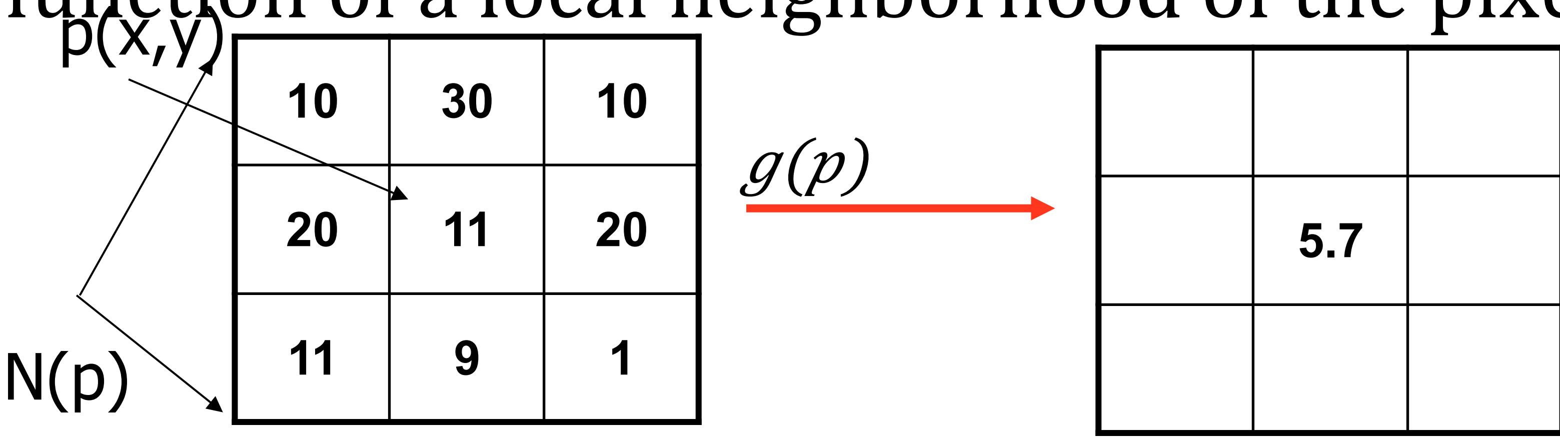
Inner product $g(x,y)=\mathbf{w}\bullet\mathbf{f}=\mathbf{w}^T\mathbf{f}$

FIGURE 3.28 The mechanics of linear spatial filtering using a 3 × 3 filter mask. The form chosen to denote the coordinates of the filter mask coefficients simplifies writing expressions for linear filtering.

- W[1:5][1:5]
- \Sum_{i=-2}^{2} \Sum_{j=-2}^{2} W(i+3,j+3)I(x+i,y+j)
- \Sum_{i=1}^{5} \Sum_{j=1}^{5} W(I,j)I(x+i-3,y+j-3)

Fundamentals of Spatial Filtering

- Modifying the pixels in an image based on some function of a local neighborhood of the pixels



g(p):

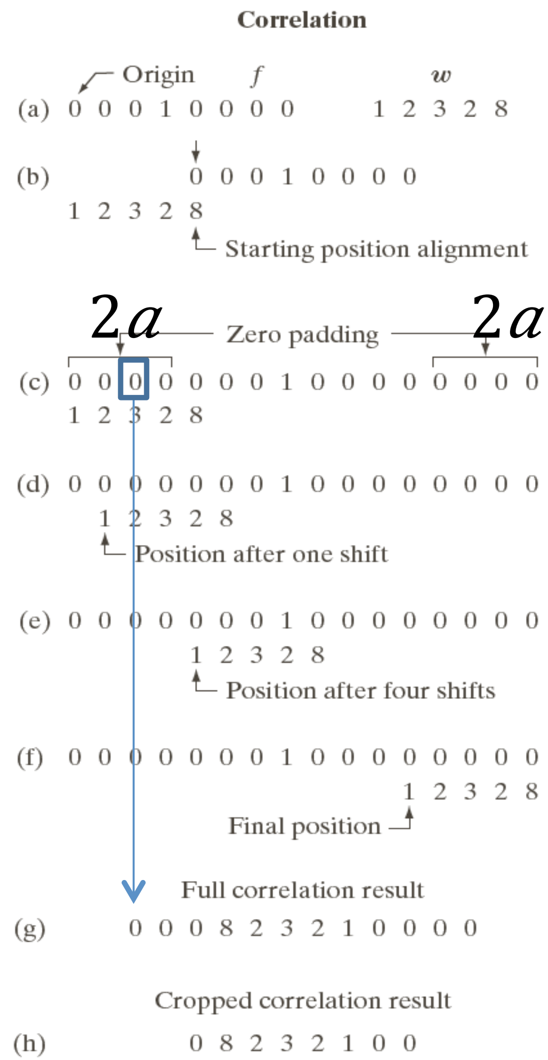
- Linear function
  - Correlation
  - Convolution
- Nonlinear function
  - Order statistic (median)

Spatial Correlation: 1D Signal

1D correlation $\sum_{s=-a}^{a} w(s) f(x+s)$

Zero-padding: add zeros on the left and right margin, respectively

- **Full correlation result** has the size of $M+2a$
- **Cropped result** has the size of M – the size of the original signal

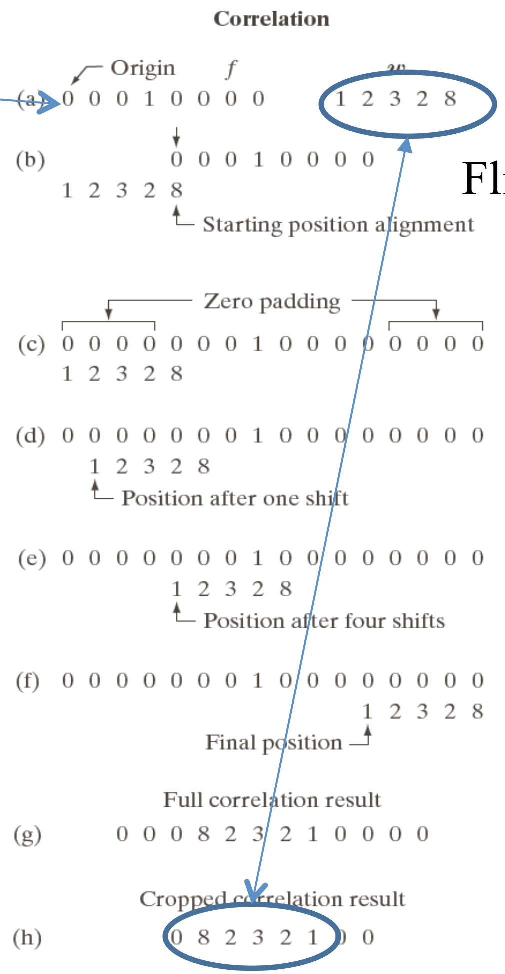


Padding

- Zero padding
- Repeat neighbor
- Repeat sequence
- Truncate

Spatial Correlation: 1D Signal

Discrete impulse



Flipped

1D correlation $\sum_{s=-a}^{a} w(s) f(x+s)$

- Full correlation result has the size of $M+2a$
- Cropped result has the size of M – the size of the original signal

The impulse response is a rotation of the filter by 180 degree

Notes from class

- T=[1 2 3 4 5]
- S=[1 2 3 4 5 6 7 8 9 … 100]
- $(I-T)^2=I^2+T^2-2I*T$

MATLAB

```
function correlation1d
S=uint8(rand(100,1)*255);
figure(1); clf; plot(S);
I=uint8(rand(1,1)*89)+1
T=S(I:I+10);
figure(2); clf; plot(T);
w=uint8(size(T,1)/2)
C=zeros(100);
for k=w:(size(S,1)-w)
    p=double(S(k-w+1:k+w-1));
    C(k)=sum(p.*double(T))/sqrt(sum(p.^2));
end
figure(3); clf;
plot(C);hold on;
plot(I+5,C(I+5),'r*');
```

Questions?