



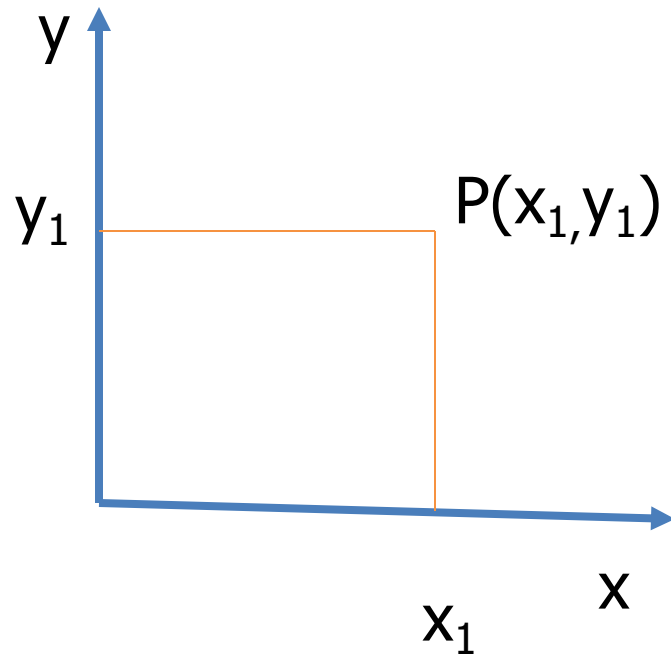
UNIVERSITY OF
SOUTH CAROLINA

CSCE 574 ROBOTICS

Coordinate Systems



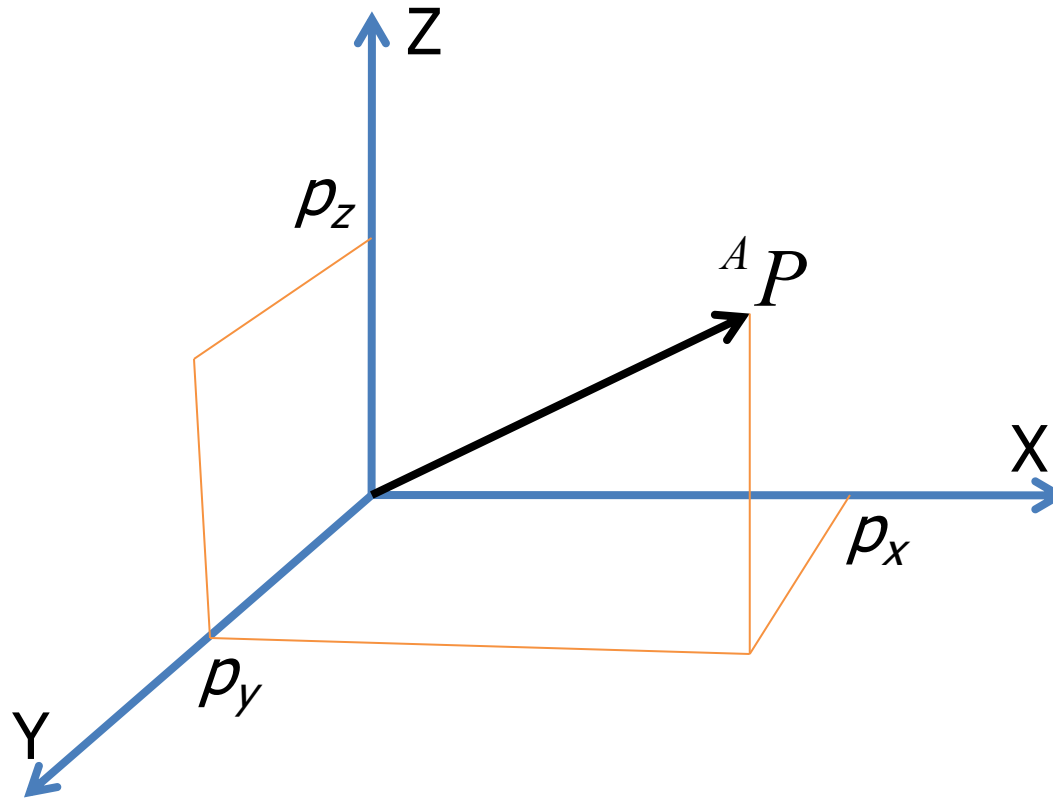
Coordinate Systems



Position Representation

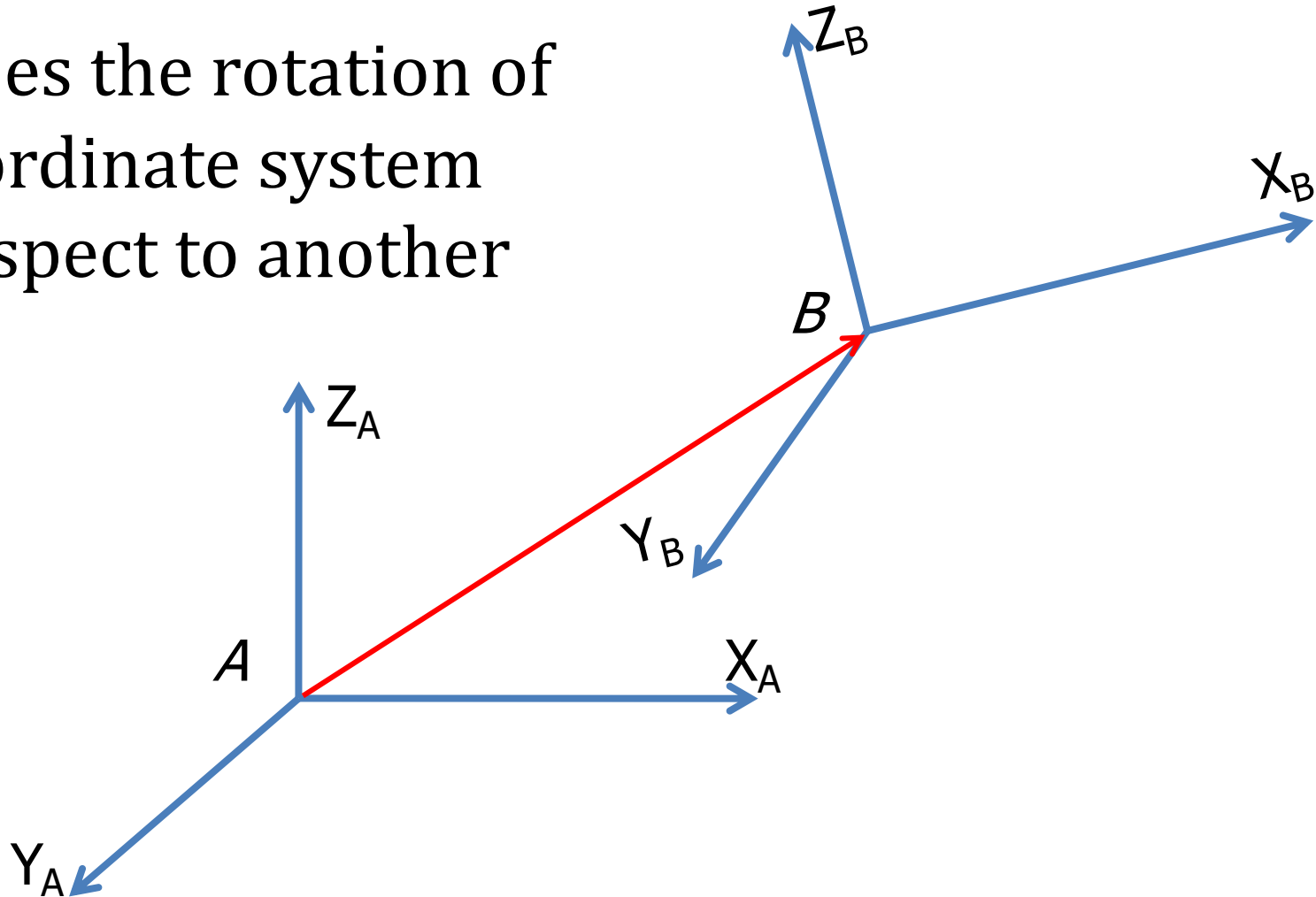
- Position representation is:

$${}^A P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



Orientation Representations

- Describes the rotation of one coordinate system with respect to another

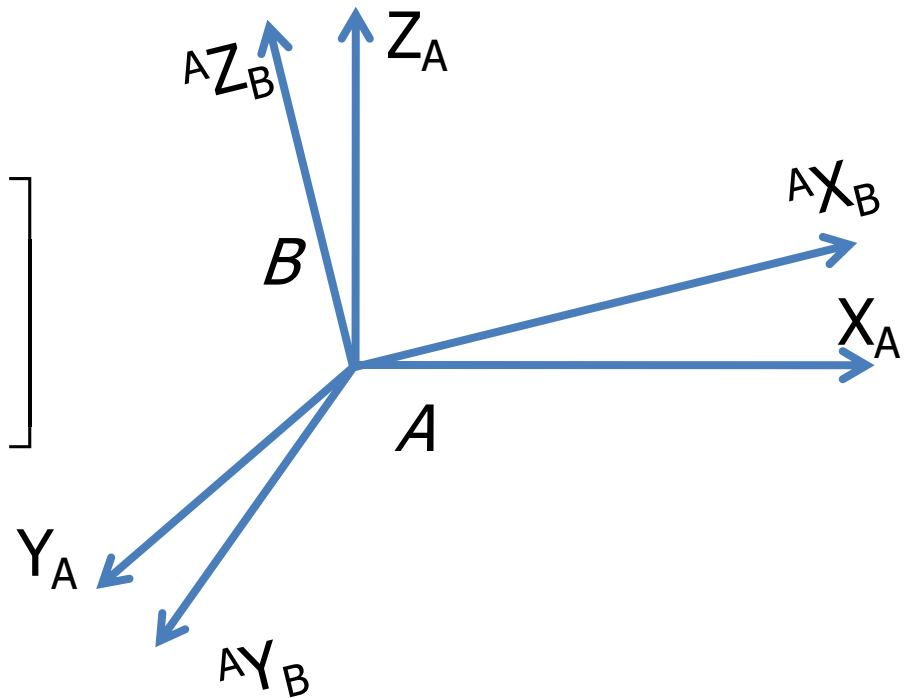


Rotation Matrix

- Write the unit vectors of B in the coordinate system of A .
- Rotation Matrix:

$${}^A_B R = \begin{bmatrix} {}^A \hat{X}_B & {}^A \hat{Y}_B & {}^A \hat{Z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$



Properties of Rotation Matrix

$${}^B R_A = {}^A R_B^T$$

$${}^A R_B^T {}^A R_B = I_3$$

$${}^A R_B = {}^B R_A^{-1} = {}^B R_A^T$$



Coordinate System Transformation

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & T \\ \mathbf{0}_{3 \times 1} & 1 \end{bmatrix}$$

where R is the rotation matrix and T is the translation vector



Rotation Matrix

- The rotation matrix consists of 9 variables, but there are many constraints. The minimum number of variables needed to describe a rotation is three.



Rotation Matrix-Single Axis

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Fixed Angles

- One simple method is to perform three rotations about the axis of the original coordinate frame:
 - X-Y-Z fixed angles

$${}^A_B R(\theta, \phi, \psi) = R_z(\psi)R_y(\phi)R_x(\theta)$$

$$= \begin{bmatrix} \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta) - \sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta) + \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta) + \cos(\psi)\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix}$$

- There are 12 different combinations



Inverse Problem

- From a Rotation matrix find the fixed angle rotations:

$${}^A_B R(\theta, \phi, \psi) = {}^A_B R \Rightarrow$$

$$\begin{bmatrix} \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta) - \sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta) + \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta) + \cos(\psi)\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

thus :

$$\phi = A \tan 2\left(-r_{31}, \sqrt{(r_{11}^2 + r_{21}^2)}\right)$$

$$\psi = A \tan 2\left(\frac{r_{21}}{\cos(\phi)}, \frac{r_{11}}{\cos(\phi)}\right)$$

$$\theta = A \tan 2\left(\frac{r_{32}}{\cos(\phi)}, \frac{r_{33}}{\cos(\phi)}\right)$$



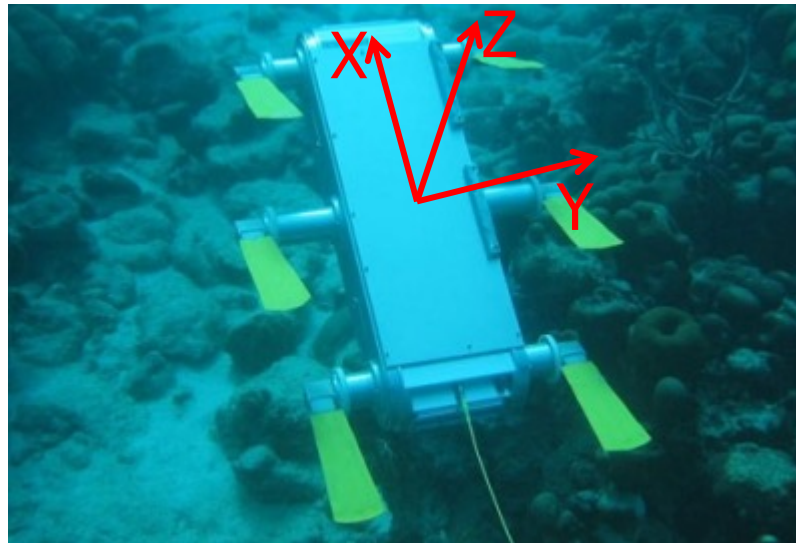
Euler Angles

- **ZYX**: Starting with the two frames aligned, first rotate about the Z_B axis, then by the Y_B axis and then by the X_B axis. The results are the same as with using XYZ fixed angle rotation.
- There are 12 different combination of Euler Angle representations



Euler Angles

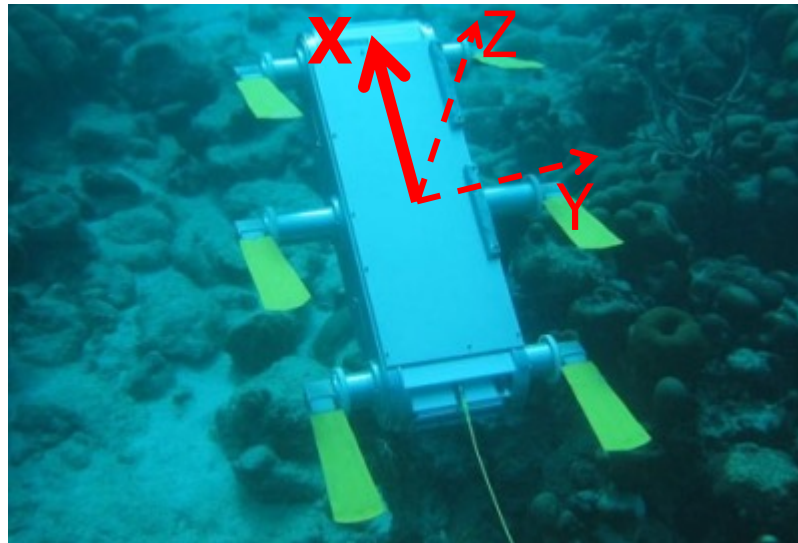
- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw



Euler Angles

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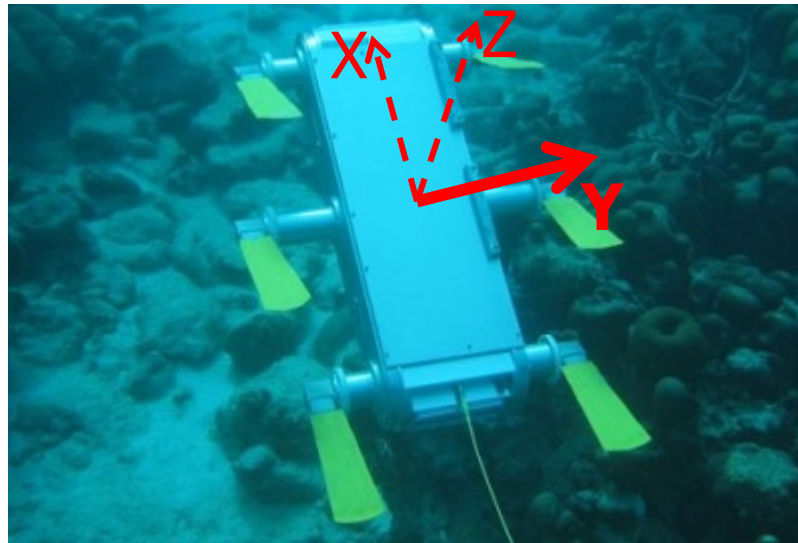
Roll



Euler Angles

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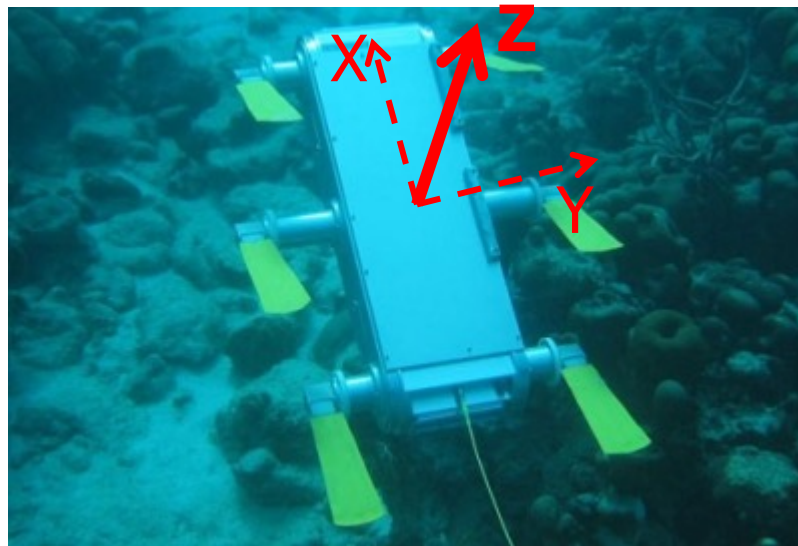
Pitch



Euler Angles

- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

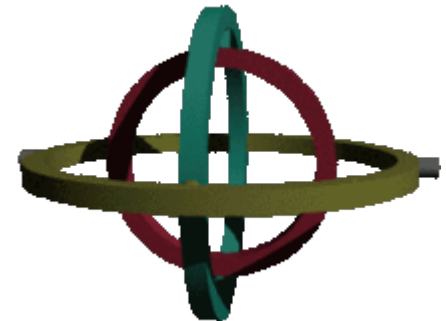
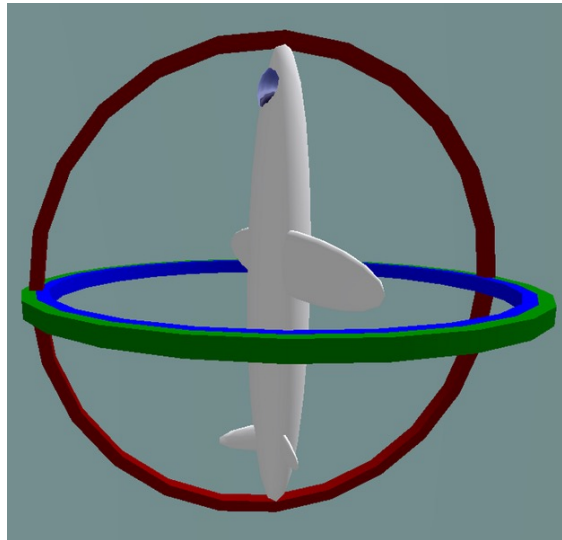
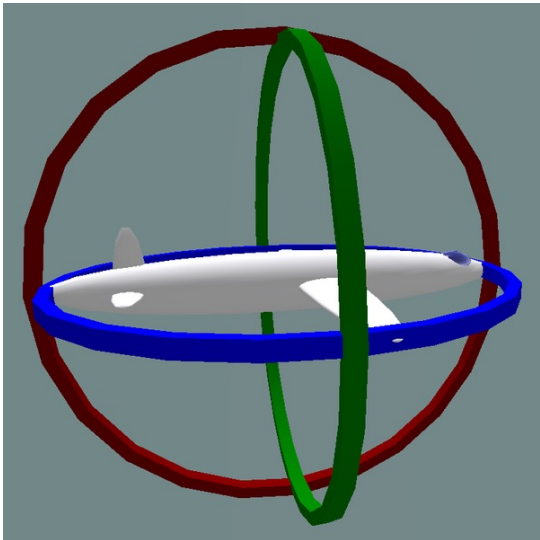
Yaw



Euler Angle concerns: Gimbal Lock

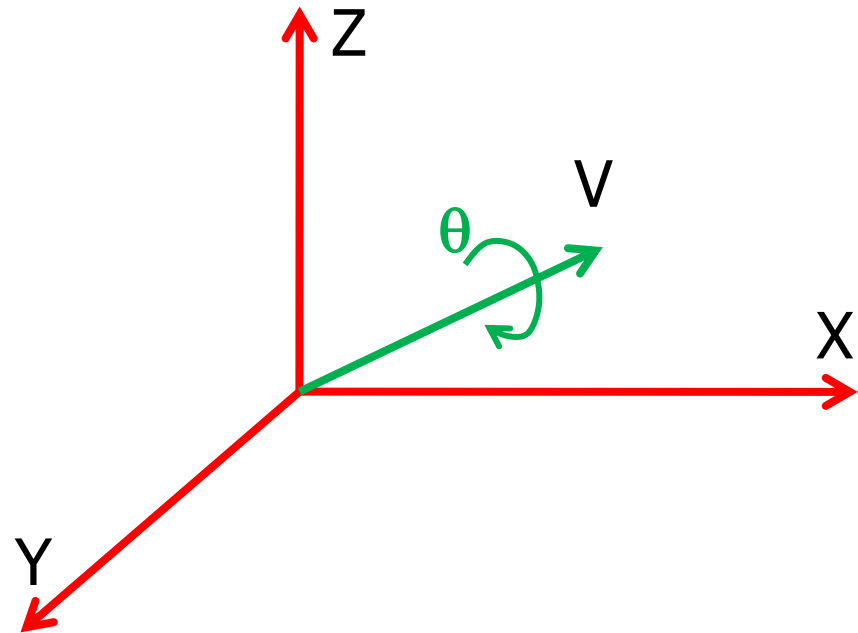
Using the **ZYZ** convention

- $(90^\circ, 45^\circ, -105^\circ) \equiv (-270^\circ, -315^\circ, 255^\circ)$ multiples of 360°
- $(72^\circ, 0^\circ, 0^\circ) \equiv (40^\circ, 0^\circ, 32^\circ)$ singular alignment (Gimbal lock)
- $(45^\circ, 60^\circ, -30^\circ) \equiv (-135^\circ, -60^\circ, 150^\circ)$ bistable flip



Axis-Angle Representation

- Represent an arbitrary rotation as a combination of a vector and an angle



Quaternions

- Are similar to axis-angle representation
- Two formulations
 - Classical
 - Based on JPL's standards
 - W. G. Breckenridge, "Quaternions - Proposed Standard Conventions," JPL, Tech. Rep. INTEROFFICE MEMORANDUM IOM 343-79-1199, 1999.
- Avoids Gimbal lock
- See also: M. D. Shuster, "A survey of attitude representations," Journal of the Astronautical Sciences, vol. 41, no. 4, pp. 439–517, Oct.–Dec. 1993.



Quaternions

	Classic notation	JPL-based
	$\bar{q} = q_4 + q_1i + q_2j + q_3k$	$\bar{q} = q_4 + q_1i + q_2j + q_3k$
	$i^2 = j^2 = k^2 = ijk = -1$	$i^2 = j^2 = k^2 = -1$
	$ij = -ji = k, jk = -kj = i, ki = -ik = j$	$-ij = ji = k, -jk = kj = i, -ki = ik = j$
Vector Notation	$\bar{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, q_0 = \cos(\theta/2), \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \sin(\theta/2)\cos(\beta_x) \\ \sin(\theta/2)\cos(\beta_y) \\ \sin(\theta/2)\cos(\beta_z) \end{bmatrix}$	$\bar{q} = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} k_x \sin(\theta/2) \\ k_y \sin(\theta/2) \\ k_z \sin(\theta/2) \end{bmatrix}, q_4 = \cos(\theta/2)$
		$\ \bar{q}\ = 1, \bar{q} \otimes \bar{p}, \mathbf{q} \times \mathbf{p}, \bar{q}_I, \lfloor \mathbf{q} \times \rfloor$

See also: N. Trawny and S. I. Roumeliotis, "Indirect Kalman Filter for 3D Attitude Estimation," University of Minnesota, Dept. of Comp. Sci. & Eng., Tech. Rep. 2005-002, March 2005.



Coordinate frames on PR2

