Today’s Agenda

- **Fourier Transform**
- **Discrete Time Fourier Transform**
- **Discrete Fourier Transform**

Recall: Fourier Series

***f*(*t*) is a continuous function with period *T*, we have**

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{j2\pi nt}{T}}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-\frac{j2\pi nt}{T}} dt, \;\; n = 0, \pm 1, \pm 2, \ldots$$

Recall: Fourier Transform in 1D

Spatial domain → Frequency domain

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \qquad \text{Forward transform}$$

Frequency domain → Spatial domain

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu \qquad \text{Inverse transform}$$

Fourier transform pair

Basic Properties of FT

Linearity $h(t) = af(t) + bg(t) \leftrightarrow H(\mu) = aF(\mu) + bG(\mu)$

Translation $h(t) = f(t - t_0) \leftrightarrow H(\mu) = e^{-j2\pi t_0 \mu} F(\mu)$

Modulation $h(t) = e^{j2\pi\mu_0 t} f(t) \leftrightarrow H(\mu) = F(\mu - \mu_0)$

Scaling $h(t) = f(at) \leftrightarrow H(\mu) = \frac{1}{|a|} F(\frac{\mu}{a})$

Conjugation $h(t) = f^*(t) \leftrightarrow H(\mu) = F^*(-\mu)$

Symmetry $f(t) \leftrightarrow F(\mu) \Rightarrow F(t) \leftrightarrow f(-\mu)$

FT of an Impulse

$$\delta(t) \leftrightarrow F(\mu) = 1$$

$$\delta(t - t_0) \leftrightarrow F(\mu) = e^{-j2\pi\mu t_0}$$

FT of an Impulse

$$e^{j2\pi t_0 t} \leftrightarrow ?$$

$$F\left(e^{j2\pi t_0 t}\right) = \delta(\mu - t_0)$$

Symmetry property

$$f(t) \leftrightarrow F(\mu) \Rightarrow F(t) \leftrightarrow f(-\mu)$$

$$\delta(t - t_0) \leftrightarrow F(\mu) = e^{-j2\pi u t_0}$$

$$F\left(e^{-j2\pi t_0 t}\right) = \delta(-\mu - t_0)$$

Scaling property

$$h(t) = f(at) \leftrightarrow H(\mu) = \frac{1}{|a|} F(\frac{\mu}{a})$$

$$F\left(e^{j2\pi t_0 t}\right) = \delta(\mu - t_0)$$

Discrete Impulses and Sifting Property

Unit impulse

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \sum_{x=-\infty}^{+\infty} \delta(x) = 1$$

Sifting property

$$\sum_{x=-\infty}^{\infty} \delta(x) g(x) = g(0)$$

$$\sum_{x=-\infty}^{\infty} \delta(x - x_0) g(x) = g(x_0)$$

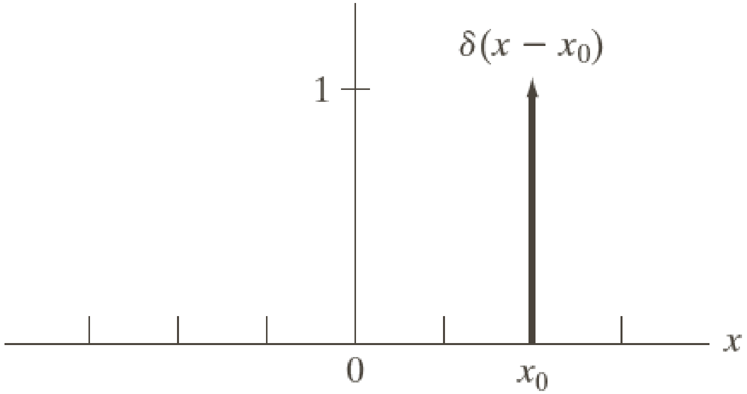


FIGURE 4.2
A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Impulse Train

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

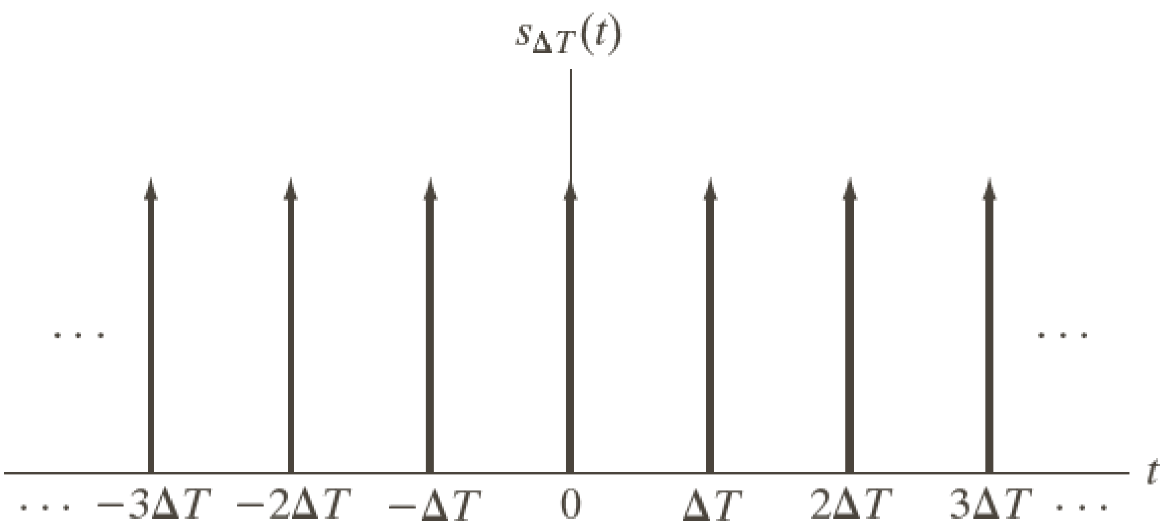
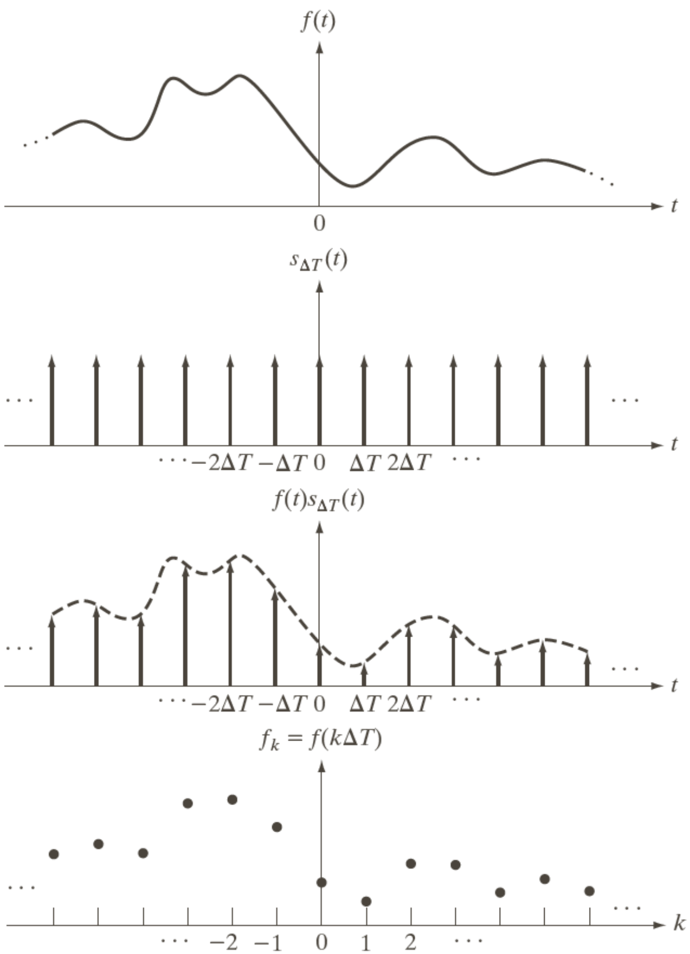


FIGURE 4.3 An impulse train.

Sampling in Spatial Domain



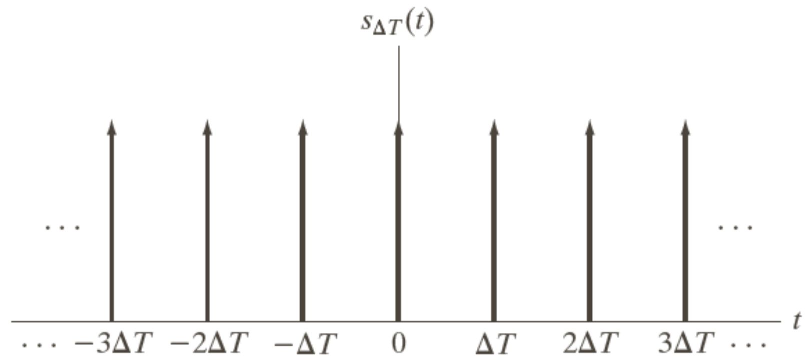
$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta T)$$

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} f(k\Delta T)$$

$\frac{1}{\Delta T}$ is the sampling rate

FT of an Impulse Train



$$s_{\Delta T}(t) \leftrightarrow ?$$

Fourier series of an impulse train $\quad s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$

where $\quad c_n = \frac{1}{\Delta T}\int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} S_{\Delta T}(t) e^{-\frac{j2\pi n t}{\Delta T}} dt \quad \Rightarrow \quad c_n = \frac{1}{\Delta T}$

$$\Rightarrow \quad s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

FT of an Impulse and Impulse Train

$$e^{j2\pi t_0 t} \leftrightarrow \delta(\mu - t_0)$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{\Delta T} t}$$

Let $t_0 = \frac{n}{\Delta T}$

$$S(\mu) = F(s_{\Delta T}(t)) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

FT of an impulse train is an impulse train in frequency domain

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \Leftrightarrow \quad S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

Convolution

Convolution in the spatial domain

Recall 1D convolution $\sum_{s=-a}^{a} w(s) f(x-s)$

Continuous $f(t) \otimes h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$

convolution

What is its FT? $F(\mu)H(\mu)$

How to prove it?

Convolution

$$f(t) \otimes h(t) = F(\mu)H(\mu)$$

$$f(t)h(t) = F(\mu) \otimes H(\mu)$$

Note: the image and the kernel should be the same size

Sampling in Spatial Domain

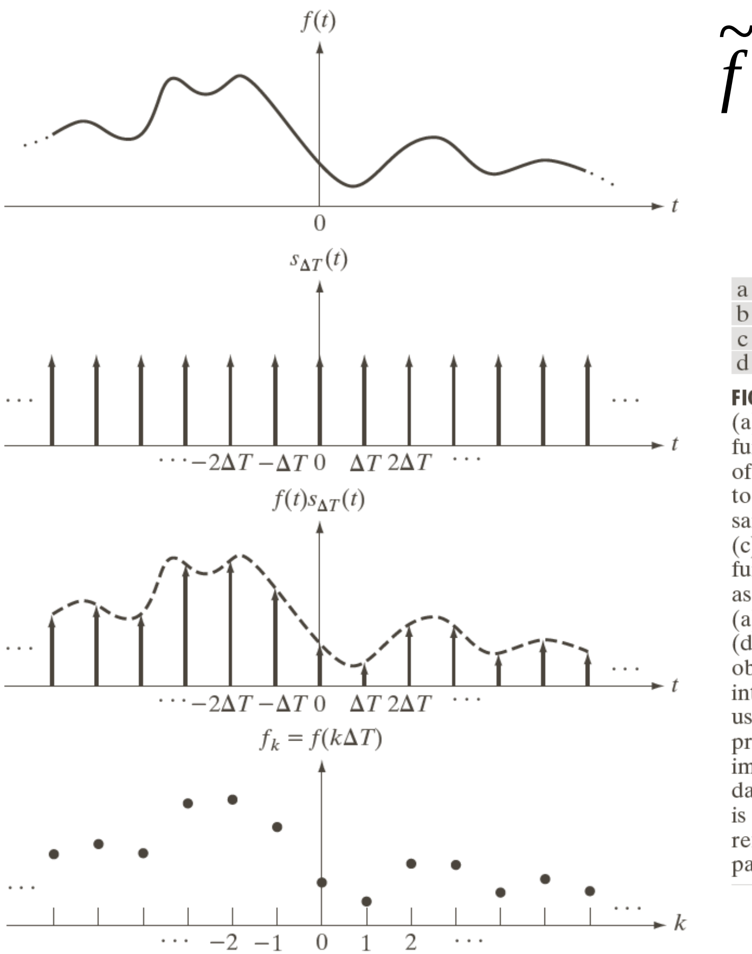


FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} f(k\Delta T)$$

$\dfrac{1}{\Delta T}$ is the sample rate

Sampling in Spatial Domain

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

What is Fourier Transform of $\tilde{f}(t)$?

FT of $S_{\Delta T}(t)$

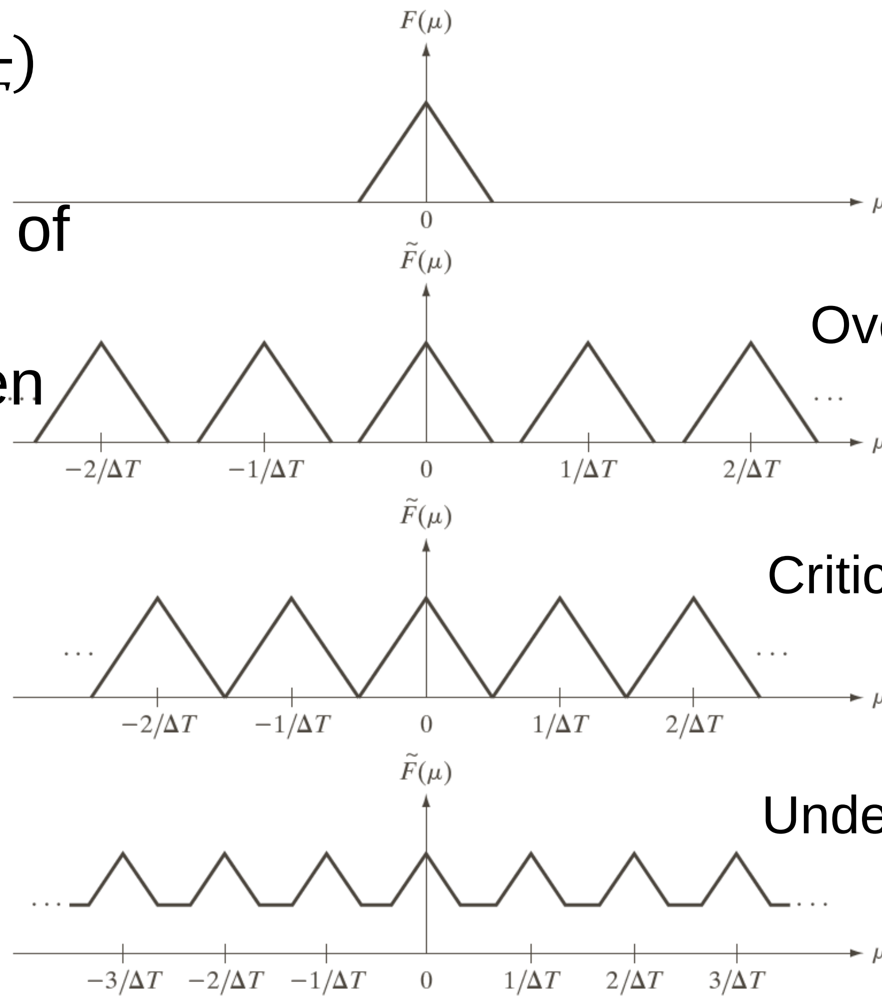
$$\tilde{F}(\mu) = F(\mu) \otimes S(\mu) = \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty} F(\mu - \frac{k}{\Delta T})$$

FT of $f(t)$

Sampling in Frequency Domain

$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty} F(\mu - \frac{k}{\Delta T})$ is

- a summation of copies of $F(\mu)$
- The separation between two copies is $\frac{1}{\Delta T}$
- Periodic
- Continuous



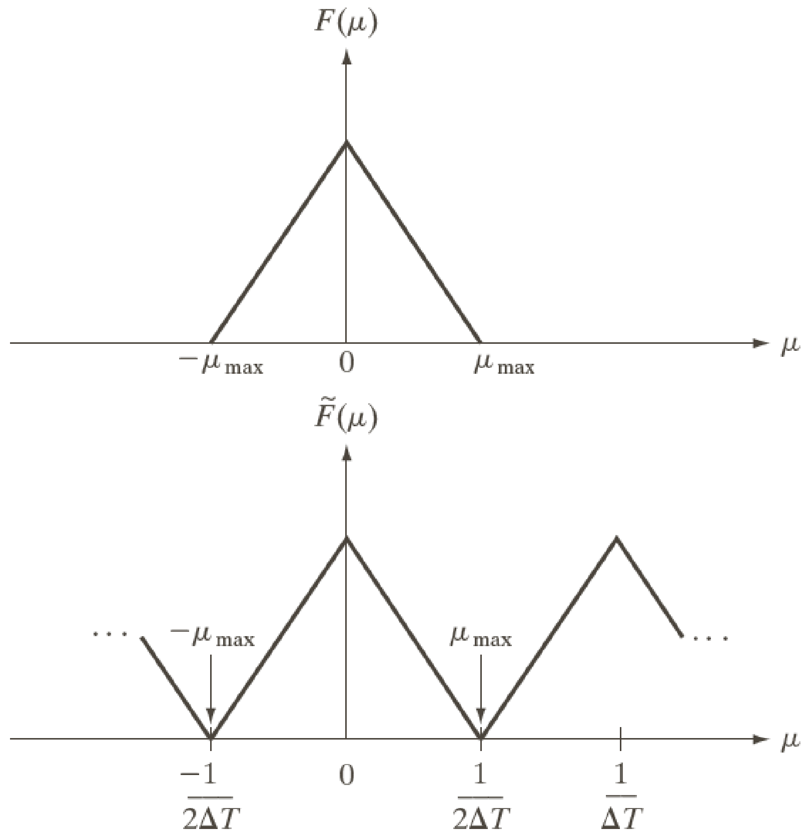
$F(u)$

Over-sampling

Critical-sampling

Under-sampling

Critical Sampling



a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

$\tilde{F}(\mu)$ is periodic ⇨ One period of $\tilde{F}(\mu)$ can represent $\tilde{F}(\mu)$

Original signal can be reconstructed perfectly from sampled data

Reconstruction and Sampling Theorem

$$f(t) \leftrightarrow F(\mu) = H(\mu)\tilde{F}(\mu)$$

$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$

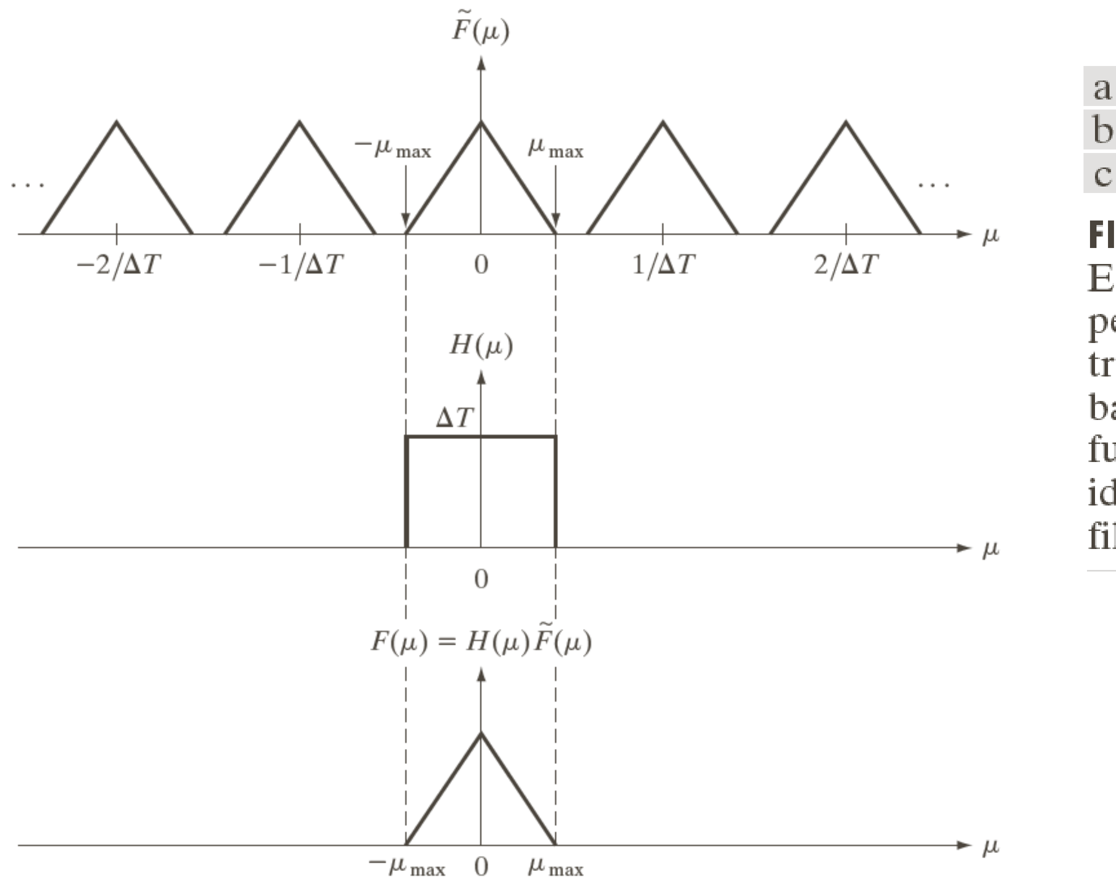
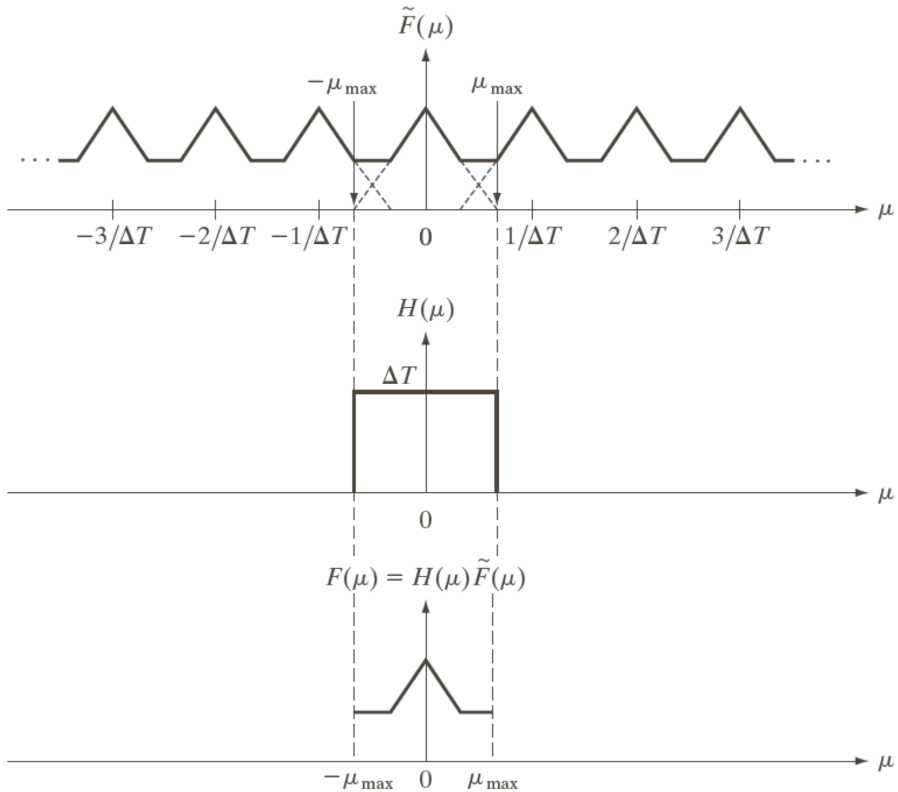


FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

Aliasing – Under Sampling



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Aliasing in Images. Have you ever come across an image like… | by Rishabh Gupta | Medium

Aliasing

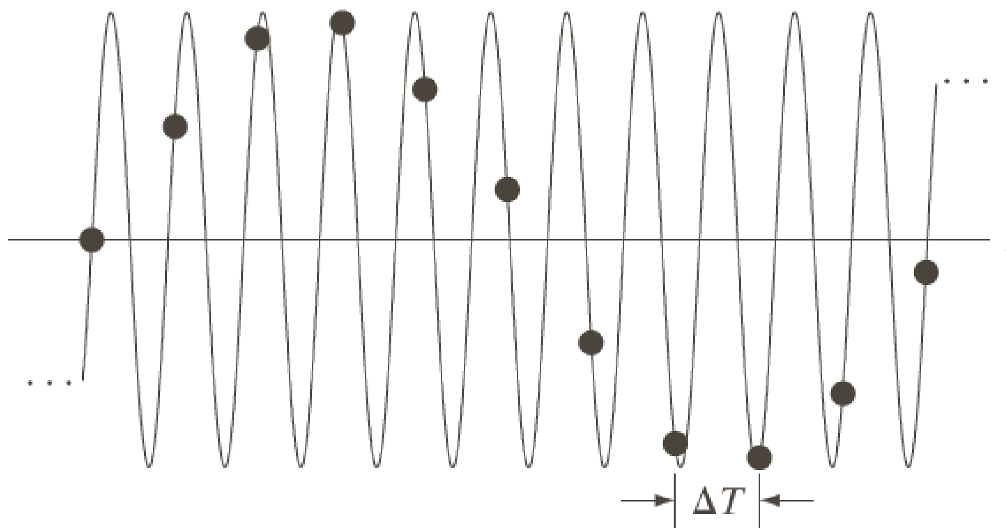


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Discrete-Time Fourier Transform

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

Discrete data -> interval has discrete-time

Discrete-Time Fourier Transform

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

Definition of FT

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t} dt$$

$$= \sum_{k=-\infty}^{\infty} f(k\Delta T)\, e^{-j2\pi\mu k\Delta T}$$

Discrete-Time Fourier Transform

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

Discrete-Time Fourier Transform

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty} F(\mu - \frac{k}{\Delta T}) = \sum_{k=-\infty}^{\infty} f(k\Delta T)\, e^{-j2\pi\mu k\Delta T}$$

$\tilde{F}(\mu)$ is continuous → Difficult to implement in DSP applications

Sample $\tilde{F}(\mu)$ in one period with M equally spaced samples ⇨ Discrete Fourier Transform

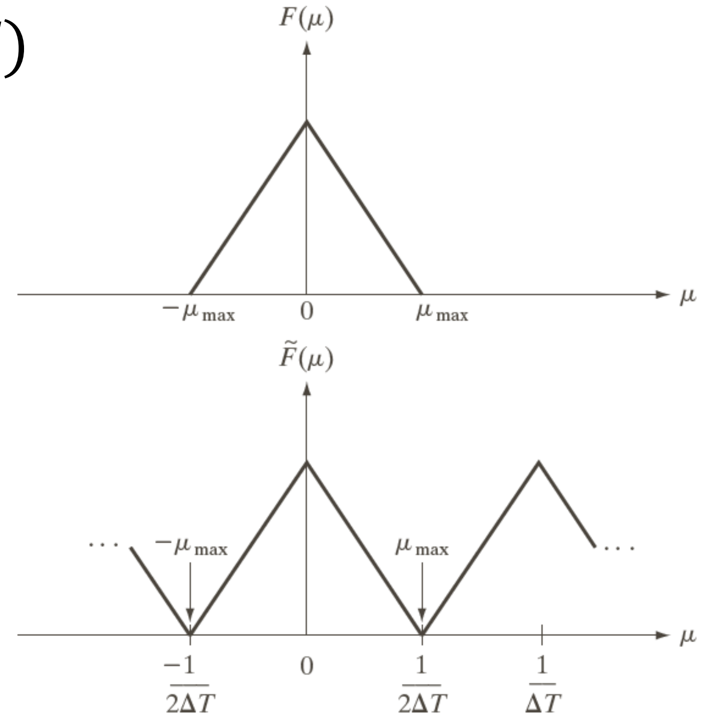
Note that

Total span of one period in spatial domain: T

1 unit in spatial domain: $\Delta T = \frac{1}{M} T$

Total *M* units in the frequency domain: $1/\Delta T$

1 unit in frequency domain: $\Delta \mu = 1/(M\Delta T)$



Discrete Fourier Transform (DFT)

coefficients of a combination of sinusoids

a finite sequence of equally spaced samples

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi xu/M}$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}$$

$$x, u = 0,1,2,\ldots,M-1$$

- $1/M$ is the sampling interval
- u is an integer → the frequency is an integer multiplier of $\frac{2\pi}{M}$
- Both input & output are finite

Discrete FT (DFT)

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

$$x, u = 0,1,2,\ldots, M - 1$$

- DFT is periodic with a period of M
- Both input & output are finite

DFT is important for digital signal processing and digital image processing

Discrete FT (DFT)

Circular Convolution:

$$f(x) * h(x) = \sum_{m=0}^{M-1} f(m) h(x-m)$$

- The convolution is periodic

Reading Assignments

Chapter 4.3 – 4.11