Bayesian Networks and Decision Graphs

Chapter 1

Two perspectives on probability theory

In many domains, the probability of an outcome is interpreted as a relative frequency:

- The probability of getting a three by throwing a six-sided die is $1/6$.

However, we often talk about the probability of an event without being able to specify a frequency for it:

- What is the probability that Denmark wins the world cup in 2010?

Such probabilities are called subjective probabilities

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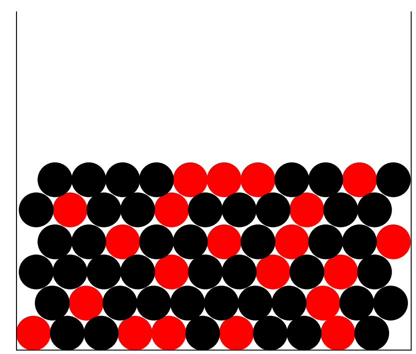
Possible interpretation:

- I receive Dkr 1000 if Denmark wins.
- If I draw a red ball I receive Dkr 1000.



Basic probability axioms

The set of possible outcomes of an “experiment” is called the sample space $\mathcal{S}$:

- Throwing a six sided die: $\{1, 2, 3, 4, 5, 6\}$.
- Will Denmark win the world cup: {yes,no}.
- The values in a deck of cards: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$.

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An event $\mathcal{E}$ is a subset of the sample space:

- The event that we will get an even number when throwing a die: $\{2, 4, 6\}$.
- The event that Denmark wins: $\{$yes$\}$.
- The event that we will get a 6 or below when drawing a card: $\{2, 3, 4, 5, 6\}$.

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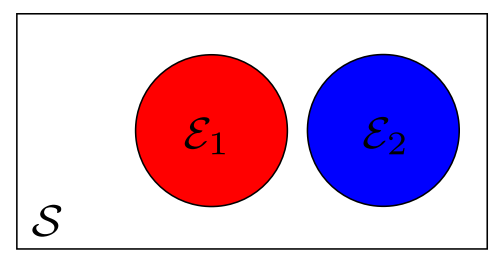
We measure our uncertainty about an experiment by assigning probabilities to each event. The probabilities must obey the following axioms:

- $P(\mathcal{S}) = 1$.
- For all events $\mathcal{E}$ it holds that $P(\mathcal{E}) \geq 0$.
- If $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$, then $P(\mathcal{E}_1 \cup \mathcal{E}_2) = P(\mathcal{E}_1) + P(\mathcal{E}_2)$.



Conditional probabilities

Every probability is conditioned on a context. For example, if we throw a dice:

"$P(\{\text{six}\}) = \frac{1}{6}$" = "$P(\text{six}|\text{symmetric dice}) = \frac{1}{6}$".

In general, if $\mathcal{A}$ and $\mathcal{B}$ are events and $P(\mathcal{A}|\mathcal{B}) = x$, then:

"In the context of $\mathcal{B}$ we have that $P(\mathcal{A}) = x$"

Note: It is not "whenever $\mathcal{B}$ we have $P(\mathcal{A}) = x$", but rather: if $\mathcal{B}$ and everything else known is irrelevant to $\mathcal{A}$, then $P(\mathcal{A}) = x$.

Definition: For two events $\mathcal{A}$ and $\mathcal{B}$ we have:

$$P(\mathcal{A}|\mathcal{B}) = \frac{P(\mathcal{A} \cap \mathcal{B})}{P(\mathcal{B})}$$

Example:

$$P(\mathcal{A} = \{4\}|\mathcal{B} = \{2,4,6\}) = \frac{P(\mathcal{A} \cap \mathcal{B} = \{4\})}{P(\mathcal{B} = \{2,4,6\})} = \frac{1/6}{3/6} = \frac{1}{3}.$$

Basic probability calculus: the fundamental rule

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be events.

The fundamental rule: $P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B})$.

The fundamental rule, conditioned: $P(\mathcal{A} \cap \mathcal{B}|\mathcal{C}) = P(\mathcal{A}|\mathcal{B} \cap \mathcal{C})P(\mathcal{B}|\mathcal{C})$.

Proof: Derived directly from the definition of conditional probability.

Basic probability calculus: Bayes' rule

Bayes rule:

$$P(\mathcal{B}|\mathcal{A}) = \frac{P(\mathcal{A}|\mathcal{B})P(\mathcal{B})}{P(\mathcal{A})}$$

Proof:

$$P(\mathcal{B}|\mathcal{A})P(\mathcal{A}) = P(\mathcal{B} \cap \mathcal{A}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B})$$

Bayes rule, conditioned:

$$P(\mathcal{B}|\mathcal{A} \cap \mathcal{C}) = \frac{P(\mathcal{A}|\mathcal{B} \cap \mathcal{C})P(\mathcal{B}|\mathcal{C})}{P(\mathcal{A}|\mathcal{C})}$$

Example: We have two diseases $\mathcal{A}_1$ and $\mathcal{A}_2$ that are the only diseases that can cause the symptoms $\mathcal{B}$. If

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are equally likely ($P(\mathcal{A}_1) = P(\mathcal{A}_2)$)
- $P(\mathcal{B}|\mathcal{A}_1) = 0.9$
- $P(\mathcal{B}|\mathcal{A}_2) = 0.3$

what are then the probabilities $P(\mathcal{A}_1|\mathcal{B})$ and $P(\mathcal{A}_2|\mathcal{B})$?

Basic probability calculus

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be events.

Conditional probability: $P(\mathcal{A}|\mathcal{B}) = \frac{P(\mathcal{A} \cap \mathcal{B})}{P(\mathcal{B})}$

The fundamental rule: $P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B})$.

The conditional fundamental rule: $P(\mathcal{A} \cap \mathcal{B}|\mathcal{C}) = P(\mathcal{A}|\mathcal{B} \cap \mathcal{C})P(\mathcal{B}|\mathcal{C})$.

Basic probability calculus

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be events.

Conditional probability: $P(\mathcal{A}|\mathcal{B}) = \frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})}$

The fundamental rule: $P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B})$.

The conditional fundamental rule: $P(\mathcal{A} \cap \mathcal{B}|\mathcal{C}) = P(\mathcal{A}|\mathcal{B} \cap \mathcal{C})P(\mathcal{B}|\mathcal{C})$.

Bayes rule: $P(\mathcal{B}|\mathcal{A}) = \frac{P(\mathcal{A}|\mathcal{B})P(\mathcal{B})}{P(\mathcal{A})}$.

Proof: $P(\mathcal{B}|\mathcal{A})P(\mathcal{A}) = P(\mathcal{B} \cap \mathcal{A}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B})$.

Bayes rule, conditioned: $P(\mathcal{B}|\mathcal{A} \cap \mathcal{C}) = \frac{P(\mathcal{A}|\mathcal{B}\cap\mathcal{C})P(\mathcal{B}|\mathcal{C})}{P(\mathcal{A}|\mathcal{C})}$.

Conditional independence: If $P(\mathcal{A}|\mathcal{B} \cap \mathcal{C}) = P(\mathcal{A}|\mathcal{C})$ then $P(\mathcal{A} \cap \mathcal{B}|\mathcal{C}) = P(\mathcal{A}|\mathcal{C}) \cdot P(\mathcal{B}|\mathcal{C})$.

Probability calculus for variables

A is a variable with states $a_1, \ldots, a_n$; B is a variable with states $b_1, \ldots, b_m$.

$P(A) = (x_1, \ldots, x_n)$ is a probability distribution ; $x_i \geq 0$;$\sum_{i=1}^n x_i = 1$ ($\sum_A P(A) = 1$).

$P(A|B)$ is a $n \times m$ table containing the numbers $P(a_i|b_j)$.

Note: $\sum_A P(A|b_j) = 1$ for all b_j.

		B		
		b_1	b_2	b_3
A	a_1	0.4	0.3	0.6
	a_2	0.6	0.7	0.4

$P(A, B)$ is a $n \times m$ table too; $\sum_{A,B} P(A, B) = 1$.

		B		
		b_1	b_2	b_3
A	a_1	0.16	0.12	0.12
	a_2	0.24	0.28	0.08

The fundamental rule for variables

$P(A|B)P(B)$: $n \times m$ multiplications $P(a_i|b_j)P(b_j) = P(a_i, b_j)$

	b_1	b_2	b_3
a_1	0.4	0.3	0.6
a_2	0.6	0.7	0.4

$P(A|B)$

b_1	b_2	b_3
0.4	0.4	0.2

$P(B)$

=

	b_1	b_2	b_3
a_1	0.16	0.12	0.12
a_2	0.24	0.28	0.08

$P(A, B)$

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$P(A|B)$

b_1	b_2	b_3
0.4	0.4	0.2

$P(B)$

=

	b_1	b_2	b_3
a_1	0.16	0.12	0.12
a_2	0.24	0.28	0.08

$P(A, B)$

A is independent of B given C if $P(A|B, C) = P(A|C)$.

	b_1	b_2	b_3
c_1	$(0.4, 0.6)$	$(0.4, 0.6)$	$(0.4, 0.6)$
c_2	$(0.7, 0.3)$	$(0.7, 0.3)$	$(0.7, 0.3)$

$P(A|B, C)$

=

	a_1	a_2
c_1	0.4	0.6
c_2	0.7	0.3

$P(A|C)$

Marginalization

We have $P(A, B)$ and we need $P(A)$.

	b_1	b_2	b_3		
a_1	0.16	0.12	0.12	$\rightarrow$	0.4
a_2	0.24	0.28	0.08	$\rightarrow$	0.6

B is marginalized out of $P(A, B)$:

$$
\begin{aligned}
"A = a_1" &= ("A = a_1" \wedge "B = b_1") \vee ("A = a_1" \wedge "B = b_2") \vee ("A = a_1" \wedge "B = b_3") \\
&= 0.16 + 0.12 + 0.12 = 0.4 \\
"A = a_2" &= ("A = a_2" \wedge "B = b_1") \vee ("A = a_2" \wedge "B = b_2") \vee ("A = a_2" \wedge "B = b_3") \\
&= 0.24 + 0.28 + 0.08 = 0.6
\end{aligned}
$$

Notation: $P(A) = \sum_B P(A, B)$

Notation

A potential ϕ is a table of real numbers over a set of variables, dom(ϕ).

A table of probabilities is a probability potential.

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A table of probabilities is a probability potential.

Multiplication

	b_1	b_2
a_1	2	1
a_2	3	4

	b_1	b_2
a_1	1	2
a_2	5	6

=

	b_1	b_2
a_1		
a_2		

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Multiplication

	b_1	b_2
a_1	2	1
a_2	3	4

	b_1	b_2
a_1	1	2
a_2	5	6

=

	b_1	b_2
a_1	2	
a_2		

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A potential ϕ is a table of real numbers over a set of variables, dom(ϕ).

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Multiplication

	b_1	b_2
a_1	2	1
a_2	3	4

	b_1	b_2
a_1	1	2
a_2	5	6

=

	b_1	b_2
a_1	2	2
a_2		

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Multiplication

	b_1	b_2
a_1	2	1
a_2	3	4

	b_1	b_2
a_1	1	2
a_2	5	6

=

	b_1	b_2
a_1	2	2
a_2	15	

Notation

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Multiplication

	b_1	b_2
a_1	2	1
a_2	3	4

	b_1	b_2
a_1	1	2
a_2	5	6

=

	b_1	b_2
a_1	2	2
a_2	15	24

Multiplication of potentials

	b_1	b_2
a_1	1	3
a_2	4	5

	b_1	b_2
c_1	6	7
c_2	8	9

=

	b_1	b_2
a_1	$(\_, \_)$	$(\_, \_)$
a_2	$(\_, \_)$	$(\_, \_)$

$\phi_1(A, B)$

$\phi_2(C, B)$

$\phi_3(A, B, C) = \phi_1(A, B) \cdot \phi_2(C, B)$

Multiplication of potentials

	b_1	b_2
a_1	1	3
a_2	4	5

	b_1	b_2
c_1	6	7
c_2	8	9

=

	b_1	b_2
a_1	$(6_{c_1}, 8_{c_2})$	$(\_, \_)$
a_2	$(\_, \_)$	$(\_, \_)$

$\phi_1(A, B)$ $\phi_2(C, B)$ $\phi_3(A, B, C) = \phi_1(A, B) \cdot \phi_2(C, B)$

Multiplication of potentials

	b_1	b_2
a_1	1	3
a_2	4	5

	b_1	b_2
c_1	6	7
c_2	8	9

=

	b_1	b_2
a_1	$(6_{c_1}, 8_{c_2})$	$(21_{c_1}, 27_{c_2})$
a_2	$(\_, \_)$	$(\_, \_)$

$\phi_1(A, B)$ $\phi_2(C, B)$ $\phi_3(A, B, C) = \phi_1(A, B) \cdot \phi_2(C, B)$

Multiplication of potentials

	b_1	b_2
a_1	1	3
a_2	4	5

$\phi_1(A, B)$

	b_1	b_2
c_1	6	7
c_2	8	9

$\phi_2(C, B)$

=

	b_1	b_2
a_1	$(6_{c_1}, 8_{c_2})$	$(21_{c_1}, 27_{c_2})$
a_2	$(24_{c_1}, 32_{c_2})$	$(\_, \_)$

$\phi_3(A, B, C) = \phi_1(A, B) \cdot \phi_2(C, B)$

Multiplication of potentials

	b_1	b_2
a_1	1	3
a_2	4	5

	b_1	b_2
c_1	6	7
c_2	8	9

=

	b_1	b_2
a_1	$(6_{c_1}, 8_{c_2})$	$(21_{c_1}, 27_{c_2})$
a_2	$(24_{c_1}, 32_{c_2})$	$(35_{c_1}, 45_{c_2})$

$\phi_1(A, B)$ $\qquad$ $\phi_2(C, B)$ $\qquad$ $\phi_3(A, B, C) = \phi_1(A, B) \cdot \phi_2(C, B)$

Marginalization of potentials

$$\sum_{B} \left(\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 2 & 3 \\ a_2 & 1 & 4 \end{array} \right) = \begin{array}{c|c} a_1 & - \\ a_2 & - \end{array}$$

Marginalization of potentials

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$$\sum_{A} \left(\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 2 & 3 \\ a_2 & 1 & 4 \end{array} \right) = \begin{array}{c|c} b_1 & - \\ b_2 & - \end{array}$$

Marginalization of potentials

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$$\sum_A \left(\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 2 & 3 \\ a_2 & 1 & 4 \end{array} \right) = \begin{array}{c|c} b_1 & 3 \\ b_2 & 7 \end{array}$$