Mike

USCIPI REPORT 530

SEMIANNUAL TECHNICAL REPORT

Covering Research Activity During the Period

1 September 1973 through 28 February 1974

William K. Pratt
Project Director
(213) 746-2694

Image Processing Institute
University of Southern California
University Park
Los Angeles, California 90007

31 March 1974

This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Eastern Test Range under Contract No. F08606-72-C-0008, ARPA Order No. 1706

3.5 Transform Domain Spectrum Interpolation

Michael N. Huhns

Quantization occurs whenever continuous physical properties are represented numerically. A quantizer is a zero-memory nonlinear device which restricts an input variable to a finite number of possible output regions. This process is irreversible and information is invariably destroyed since only the region containing the input is known at the output. However this output data can be combined with a priori knowledge about the input to reduce the amount of information lost by interpolating between the discrete outputs.

In transform image coding a block of image pixels undergoes a two dimensional transformation using a unitary transform such as the Fourier, all values in between. Expanding the determinant of $\underline{T}(n+1)$ in terms of the last row and the last column gives an expression for the determinant of $\underline{T}(N+1)$ in terms of $F(0), F(1), \ldots, F(N+1)$. Since $F(0), F(1), \ldots, F(N)$ are known, this is an expression for the determinant of $\underline{T}(N+1)$ in terms of $F(N+1)$. Choosing $F(N+1)$ to maximize the determinant of $T(N+1)$ gives a recursive algorithm to estimate $F(N+1)$ from $F(0), F(1), \ldots, F(N)$. The recursive algorithm can be used further to estimate $F(N+2)$ from $F(1), \ldots,$ $F(N)$ and the estimated value of $F(N+1)$ i.e.

$$F(j) = \sum_{k=1}^{N} A(k)F(|j-k|) \quad \text{for} \quad j = N+1, \ldots M-1 \tag{2}$$

where $A(k)$, $k = 1, \ldots, N$, are a set of fixed constants specified by matrix $\underline{T}(N+1)$.

<u>Extrapolation of Images</u> The positive extrapolation technique discussed for one-dimensional signals in the previous section can be generalized to extrapolate two-dimensional spectral density functions as well as two-dimensional Fourier transform of images. This is achieved by extending eq. (2) to functions of two variables by letting

$$F(i,j) = \sum_{k=1}^{N} \sum_{\ell=1}^{N} A(k,\ell)F(|i-\ell|, |j-k|) \qquad \text{for } i,j = 0, 1, \ldots, M-1 \tag{3}$$

where $F(i,j)$ is the two-dimensional discrete Fourier transform of the image and consists of M^2 elements. At the receiver site $(N+1)^2$ elements are available and these $(N+1)^2$ elements are used to extrapolate the missing elements prior to taking the inverse Fourier transform to obtain a reconstruction of the original image. Analogous to the one-dimensional system, the original picture is first folded along the $x = 0$ and $y = 0$ axes to generate an even two-dimensional array. This is required to make $F(i,j)$ an array of real elements. Solving eq. (3) for $A(k,\ell)$ is straightforward since $(N+1)^2$ values of $F(i,j)$ are known.

Hadamard, or Slant transform. Next, the transform coefficients are quantized and coded for transmission. Figure 1 illustrates a typical bit assignment for a zonal quantization and coding algorithm. The number of quantization levels assigned to the coefficient at coordinate (u, v) is

$$M(u, v) = 2^{b(u, v)} \qquad (1)$$

where b(u, v) denotes the bit assignment. At the receiver, the quantized coefficients are reconstructed and an inverse transformation is performed to obtain an image estimate.

If a transform coefficient is quantized to zero bits, then its restoration is equivalent to a spectrum extrapolation as outlined by Pratt [1]. Those coefficients that are quantized to two or more levels can also be restored by a technique called spectrum interpolation.

Analysis Let the N element column vector $\underline{x}$ with probability density $p_x(\underline{x})$ denote a vector of input data samples. For two-dimensional data arrays, $\underline{x}$ is formed by column scanning the data array. Each data sample is quantized into one of M output regions, denoted by D_i, i = 1, 2, . . . , M. The estimated value of $\underline{x}$ based upon the observed D_i regions is the quantizer output vector $\underline{y}_i$. The average error in this estimate is then defined as

$$\mathcal{E} = \sum_{i=1}^{M} \int_{D_i} e(\underline{x}-\underline{y}_i) p_x(\underline{x}) d\underline{x} \qquad (2)$$

where e(·) is an arbitrary error weighting criterion. The vector of estimates $\underline{y}_i$ should be chosen to minimize the average error. This choice can be determined by utilizing the principles of calculus to find the stationary points of the error surface $\mathcal{E}$ with respect to each $\underline{y}_i$. Hence

$$\frac{\partial \mathcal{E}}{\partial \underline{y}_i} = -\int_{D_i} \frac{\partial}{\partial \underline{y}_i} [e(\underline{x}-\underline{y}_i)] p_x(\underline{x}) d\underline{x} \qquad i = 1, 2, \ldots, M \qquad (3)$$

8	8	8	8	8	8	7	7	6	6	6	6	5	5	5	5
8	8	8	7	6	6	4	4	4	4	4	4	3	3	3	3
8	8	7	5	5	4	3	3	2	2	2	2	2	2	2	2
8	7	5	3	3	3	2	2	1	1	1	1	1	1	1	1
8	6	5	3	3	2	1	1	1	1	1	1	1	1	1	1
8	6	4	3	2	2	1	1	1	1	1	1	1	1	1	1
7	4	3	2	1	1	1	1	0	0	0	0	0	0	0	0
7	4	3	2	1	1	1	1	0	0	0	0	0	0	0	0
6	4	2	1	1	1	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	1	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	1	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	1	0	0	0	0	0	0	0	0	0	0
5	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0
5	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0
5	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0
5	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0

Figure 3.5-1. Typical transform domain quantizing bit assignment.

with the assumption that the error function $e(\cdot)$ is differentiable. Solving eq. (3) for the quadratic error criterion

$$e(\underline{x}-\underline{y}_i) = Tr\{(\underline{x}-\underline{y}_i)(\underline{x}-\underline{y}_i)^T\} \tag{4}$$

one obtains

$$\frac{\partial e}{\partial \underline{y}_i} = -2(\underline{x} - \underline{y}_i) \tag{5}$$

which implies that

$$\int_{D_i} (\underline{x}-\underline{y}_i)p_x(\underline{x})d\underline{x} = 0 \qquad i = 1, 2, \ldots, M \tag{6}$$

Rearrangement reveals

$$\underline{y}_i = \frac{\int_{D_i} \underline{x}p_x(\underline{x})d\underline{x}}{\int_{D_i} p_x(\underline{x})d\underline{x}} \qquad i = 1, 2, \ldots, M \tag{7}$$

or

$$\underline{y}_i = E\{\underline{x}\,|\,\underline{x}\epsilon D_i\} \tag{8}$$

This is an expression for the best nonlinear mean square estimate of $\underline{x}$, given that $\underline{x}$ lies within region D_i.

Now assume that $\underline{x}$ is distributed according to a Gaussian probability density function

$$p_x(\underline{x}) = K \exp\{-\tfrac{1}{2}\underline{x}^T \underline{C}_x^{-1} \underline{x}\} \tag{9}$$

where $\underline{C}_x$ is the covariance matrix of $\underline{x}$ and the mean is assumed to be zero. Also let

$$D_i = \{x_i\,|\,x_j \epsilon [a_j, b_j)\} \qquad j = 1, 2, \ldots, N \tag{10}$$

Then

$$\underline{y}_i = \frac{\int_{\underline{a}}^{\underline{b}} \underline{x} K \exp\{-\tfrac{1}{2}\underline{x}^T \underline{C}_x^{-1} \underline{x}\} d\underline{x}}{\int_{\underline{a}}^{\underline{b}} K \exp\{-\tfrac{1}{2}\underline{x}^T \underline{C}_x^{-1} \underline{x}\} d\underline{x}} \tag{11}$$

Curry [2] has solved this equation for finely quantized values of x_j, i.e.

$$b_j - a_j < \sigma_j \qquad j = 1, 2, \ldots, N \tag{12}$$

where σ_j is the standard deviation of the j^{th} component of $\underline{x}$. His approach is to approximate the Gaussian density by the first three terms of its Taylor series expansion about the midpoint of the region D_i. The integration can then be performed, with the result that

$$E\{\underline{x} | \underline{x} \epsilon D_i\} = (\underline{I} - \underline{\Delta} \underline{C}_x^{-1}) \frac{\underline{b}+\underline{a}}{2} \tag{13}$$

where

$$\underline{\Delta} = \left\{ \frac{(b_j - a_j)^2}{12} \delta_{kj} \right\} \qquad k, j = 1, 2, \ldots, N \tag{14}$$

An exact solution can be obtained when the components of $\underline{x}$ are uncorrelated. In this case the covariance matrix can be expressed as

$$\underline{C}_x = \{\sigma_j^2 \delta_{kj}\} \qquad k, j = 1, 2, \ldots, N \tag{15}$$

and much computation reveals that

$$\underline{y}_i = \sqrt{\frac{2}{\pi}} \begin{bmatrix} \dfrac{-\sigma_1 (e^{-b_1^2/2\sigma_1^2} - e^{-a_1^2/2\sigma_1^2})}{\operatorname{erf} \frac{b_1}{\sqrt{2}\sigma_1} - \operatorname{erf} \frac{a_1}{\sqrt{2}\sigma_1}} \\ \vdots \\ \dfrac{-\sigma_N (e^{-b_N^2/2\sigma_N^2} - e^{-a_N^2/2\sigma_N^2})}{\operatorname{erf} \frac{b_N}{\sqrt{2}\sigma_N} - \operatorname{erf} \frac{a_N}{\sqrt{2}\sigma_N}} \end{bmatrix} \tag{16}$$

Gaussian variables which had been decorrelated by means of a Karhunen-Loeve transformation and then quantized could be restored according to a minimum mean square error criterion by utilizing this last equation.

An exact analytical solution to eq. (11) also exists when an estimate of a single vector component, x_N, is desired based upon two types of information -- (a) the other components, $x_1, x_2, \ldots, x_{N-1}$, which are known completely (quantized with an infinite number of bits); (b) the quantizer output which nonlinearly specifies the interval containing x_N. To derive this, consider

$$\underline{y}_i = E\{\underline{x} | x_1 = a_1, x_2 = a_2, \ldots, x_{N-1} = a_{N-1}; a_N \le x_N < b_N\} \quad (17)$$

or

$$\underline{y}_i = \frac{\int_{D_i} \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \\ x_N \end{pmatrix} \exp\left\{ -\tfrac{1}{2}(a_1 \ldots a_{N-1} x_N) \underline{C}_x^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \\ x_N \end{pmatrix} \right\} d\underline{x}}{\int_{D_i} \exp\left\{ -\tfrac{1}{2}(a_1 \ldots a_{N-1} x_N) \underline{C}_x^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \\ x_N \end{pmatrix} \right\} d\underline{x}} \quad (18)$$

or

$$\underline{y}_i = \frac{\int_{a_N}^{b_N} \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \\ x_N \end{pmatrix} \exp\left\{ -\tfrac{1}{2}(a_1 \ldots a_{N-1} x_N) \underline{C}_x^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \\ x_N \end{pmatrix} \right\} dx_N}{\int_{a_N}^{b_N} \exp\left\{ -\tfrac{1}{2}(a_1 \ldots a_{N-1} x_N) \underline{C}_x^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \\ x_N \end{pmatrix} \right\} dx_N} \quad (19)$$

Now denote the elements of $(\underline{C}_x)^{-1}$ by

$$(\underline{C}_x)^{-1} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{pmatrix} \tag{20}$$

Then performing the one-dimensional integrations in eq. (19) yields

$$\underline{y}_i = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N-1} \\ \dfrac{-\exp\left\{-\dfrac{1}{2r_{NN}}\left(r_{NN}x_N + \sum_{j=1}^{N-1} a_j r_{jN}\right)^2\right\}\Bigg|_{a_N}^{b_N}}{\sqrt{\dfrac{\pi r_{NN}}{2}}\,\mathrm{erf}\left(\dfrac{r_{NN}x_N + \sum_{j=1}^{N-1} a_j r_{jN}}{\sqrt{2r_{NN}}}\right)\Bigg|_{a_N}^{b_N}} - \dfrac{1}{r_{NN}}\sum_{j=1}^{N-1} a_j r_{jN} \end{pmatrix} \tag{21}$$

If x_N is quantized to an infinite number of bits, then $y_i^N = a_N = b_N$, as expected. If x_N is quantized to zero bits, its interval is the real line ($-a_N = b_N = \infty$), and then its estimate, y_i^N, is

$$y_i^N = -\frac{1}{r_{NN}}\sum_{j=1}^{N-1} a_j r_{jN} \tag{22}$$

This result is identical to that obtained by Pratt [1] in estimating an unknown spectral value based on known spectral components. However eq. (20) is a more general result in that it can be utilized to estimate components that have been quantized to any number of bits by an arbitrary quantization scheme.

Transform Domain Spectrum Interpolation The above solution is applicable to the mean square restoration of zonal coded transform samples. In this case, the transform samples have a Gaussian distribution, since each is the sum of a large number of random variables so that the central limit theorem can be invoked. These transform samples are typically quantized according to a bit assignment such as the one shown in Figure 1. For such a quantizing scheme, only eq. (16) can be utilized directly for restoration; however this equation ignores the known correlation existing between the samples. Curry's method of eq. (13) is unable to restore samples quantized to fewer than two bits. However, for greater bit assignments, it has the advantage of providing a simultaneous solution utilizing all the available information. The technique developed in eqs. (17) to (21) avoids the above difficulties, but requires a recursive solution which may be only asymptotically optimal (further analysis is expected to establish this). Therefore the best restoration, on the basis of optimality and ease of implementation, is obtained from a combination of the solutions presented above and must be adapted to the particular quantizer used. This technique will soon be applied to zonal transform coded images. It is anticipated that the resultant image will have a lower mean square error and improved subjective quality.

References

1. W. K. Pratt, "Transform Image Coding Spectrum Extrapolation," Proc. Seventh Hawaii International Conference on System Sciences, January, 1974, pp. 7-9.

2. R. E. Curry, Estimation and Control with Quantized Measurements, The M.I.T. Press, Cambridge, Massachusetts 1970.

3. Variable Rate Image Coding for Sources with Unknown Probabilities