Language nonregularity

1) Myhill-Nerode Thm
2) Pumping lemma (+ closure properties)

Def: Fix alphabet Σ. Let $L \subseteq \Sigma^*$ be any language. For $w \in \Sigma^*$, we define the tail language

$$L_w := \{x \in \Sigma^* : wx \in L\} \subseteq \Sigma^*.$$

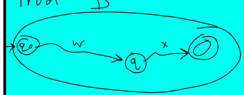
We let $\mathcal{C}_L := \{L_w : w \in \Sigma^*\}$.

Theorem (Myhill-Nerode): L is regular iff $\mathcal{C}_L$ is finite. If so, then $\mathcal{C}_L$ is the state set of the unique min DFA recognizing L.

Lemma: Suppose L is regular, & let D be a sane DFA recognizing L.
[Recall: For q a state of D, D_q is the DFA D but with start state q.]

$$\mathcal{C}_L = \{L(D_q) : q \text{ state of } D\}$$

"Proof"



$\supseteq$: Given state q, let w be string that gets from q_0 to q. Then $\forall x$

$$x \in L(D_q) \iff wx \in L(D) \; (= L)$$
$$\iff wx \text{ gets from } q_0 \text{ to an accept state.} \; //$$

$\subseteq$: Given $w \in \Sigma^*$, let q be the end state of the comp path of D on input w. Then $\forall x$

$$x \in L_w \iff wx \in L$$
$$\iff x \in L(D_q). \; //$$

Cor: If L is regular then $\mathcal{C}_L$ is finite &

$|\mathcal{C}_L| \leq$ # states of any DFA recognizing L.

Lemma: If $\mathcal{C}_L$ is finite then $\mathcal{C}_L$ is the state set of a DFA (necessarily min) recognizing L.

Proof: Let DFA

$$D := \langle \mathcal{C}_L, \Sigma, \delta, q_0, F \rangle$$

where

$$q_0 := L_\varepsilon \quad (= L)$$
$$F := \{L_w : \varepsilon \in L_w\}$$

For any $w \in \Sigma^*$ and $a \in \Sigma$, define

$$\delta(L_w, a) = L_{wa}$$

well-defined?
Need that $L_w = L_{w'}$
$\Rightarrow L_{wa} = L_{w'a} \quad \forall w, w'$

$L_w = L_{w'}$ means $\forall x$ $wx \in L$ iff $w'x \in L$.

Suppose this. Then $\forall x$

$$x \in L_{wa} \iff wax \in L$$
$$\iff ax \in L_w$$
$$\iff ax \in L_{w'} \text{ (by assumption)}$$
$$\iff x \in L_{w'a}$$

$\therefore L_{wa} = L_{w'a}$

$\therefore \delta$ is well-defined.

Remains to show that $L(D) = L$. I.e., $\forall w$ $w \in L(D) \iff w \in L$.

Induction on $|w|$.

Base case: $w = \varepsilon$

$\varepsilon \in$ ______

[Proof Deferred] $\blacksquare$

Use M-N thm to show that a language is not reg.

$\Sigma = \{0, 1\}$

$L := \{0^n 1^n : n \geq 0\}$

<u>Prop</u>: L is not regular.

<u>Proof</u>: Notice: $\forall n \geq 0$

$1^n \in L_{0^n}$ but

$1^n \notin L_{0^m}$ $m \neq n$.

$(1^n \in L_{0^{n+1}1})$

$\therefore L_\varepsilon, L_0, L_{00}, \ldots, L_{0^n},$ are pairwise distinct, so $\mathcal{C}_L$ is infinite

$\therefore L$ is not regular by M-N thm.

<u>Pumping Lemma (for regular languages)</u>: Let $L \subseteq \Sigma^*$ be a regular language. There exists $p > 0$ ("a pumping length") such that, for all $w \in L$ with $|w| \geq p$, there exist strings $x, y, z \in \Sigma^*$ such that

1) $w = xyz$
2) $|xy| \leq p$
3) $|y| > 0$ (i.e. $y \neq \varepsilon$)

and, for all $i \geq 0$

$$xy^iz \in L.$$

("pumping on y")

<u>Ex</u>:

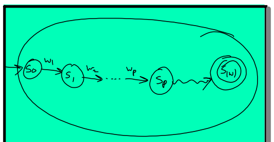
$(i=0)$	xz	
$(i=1)$	$xyz = w$	$\in L$
$(i=2)$	$xyyz$	
$(i=3)$	$xyyyz$	
$\vdots$	$\vdots$	

<u>Proof</u>: Assume L regular. Let D be a DFA recog. L.

Let p be the number of states of D.

Given a string $w \in L$ such that $|w| \geq p$, consider the comp path

$s_0, s_1, \ldots, s$



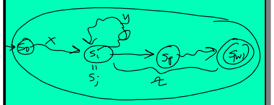
$\underbrace{s_0, \dots, s_{|w|}}$

$|w|+1$ states in the sequence

$|w| \geq p$, so $\geq p+1$ states in the sequence

Some state is repeated among $s_0, \dots, s_p$, I.e.,

$\exists\, i < j \leq p$, $s_i = s_j$



Then $w = xyz$

$|xy| = j \leq p$

$|y| > 0$ $(i \neq j)$

and: $\forall i \geq 0$,

D reading xy^iz gets to the same state $s_{|w|} \in F$

Thus $xy^iz \in L$.

<u>Def</u>: L is pumpable iff

$\exists p > 0$,

$\forall w \in L$, $|w| \geq p$

$\exists x, y, z$, $w = xyz$, $|xy| \leq p$, $|y| > 0$

$\forall i \geq 0$,

$xy^iz \in L$.

[<u>Pumping lemma</u>: Every regular language is pumpable.]

<u>Note</u>: L is not pumpable iff

$\forall p > 0$

$\exists w \in L$, $|w| \geq p$

$\forall x, y, z$ with $w = xyz$, $|xy| \leq p$, $|y| > 0$,

$\exists i \geq 0$,

$xy^iz \notin L$.

<u>Prop</u>: $L := \{0^n 1^n : n \geq 0\}$ is not pumpable.

<u>Proof</u>: Given $p > 0$,

let $w := 0^p 1^p$

$[w \in L \;\&\; |w| = 2p \geq p]$.

Given x, y, z with

$w = xyz$, $|xy| \leq p$, $|y| > 0$,

[Know that $y = 0^k$ for some $k > 0$]

let $i := 0$. Then

$xy^iz = xz = 0^{p-k} 1^p \notin L$

because $p - k \neq p$.

$\therefore$ L not pumpable //

<u>Cor</u>: L is not regular by Pump Lemma

<u>Ex</u>: $L := \{0^m 1^n : 0 \leq m \leq n\}$

<u>Prop</u>: L not pumpable.

<u>Pf</u>: Given $p > 0$,

Let $w := 0^p 1^p$

Given x, y, z, with ...

know that $y = 0^k$ (some $k > 0$)

Let $i := 2$. Then

$xyyz = 0^{p+k} 1^p \notin L$. //

($i > 1$: "pumping up")

($i = 0$: " " down")

<u>Ex</u>: $L = \{0^m 1^n : 0 \leq n \leq m\}$

<u>Prove</u>: L not pumpable.